

## COMPLEX PARALLELISABLE MANIFOLDS AND THEIR SMALL DEFORMATIONS

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### Introduction

By a complex parallelisable manifold we mean a compact complex manifold with the trivial holomorphic tangent bundle. Wang [8] showed that a complex parallelisable manifold is the quotient space of a simply connected, connected complex Lie group by one of its discrete subgroups.

It is known that if the Lie group corresponding to a parallelisable manifold is semi-simple and does not contain  $SL(2C)$  as a component, then the first Betti number vanishes and its small deformation is rigid, [2], [5], [6].

In this paper we consider the similar problems in the case where the corresponding Lie group is solvable, and obtain quite different results. We note that a simply connected, connected solvable complex Lie group is biholomorphically equivalent to  $C^n$  as a complex manifold where  $n = \dim_C G$ . If a complex parallelisable manifold has a solvable Lie group as the universal covering, it is called a complex solvable manifold.

In § 1 we summarize some known results and give three lemmas. In § 2 by numerical invariants we classify three-dimensional complex solvable manifolds into four classes III-(1), III-(2), III-(3a), III-(3b), and construct some examples in all cases.

In § 3 we construct Kuranishi families of deformations of three-dimensional complex solvable manifolds constructed in § 2. The base spaces of these Kuranishi families which are reduced complex spaces are irreducible in the cases of III-(2) and III-(3a) but reducible in for case of III-(3b), about which we shall give explicit descriptions.

For a compact complex manifold  $X$  we denote by  $\mathcal{O}$  and  $\Omega^p$  the sheaves of germs over  $X$  of holomorphic functions and  $p$ -forms respectively. Recall  $h^{p,q} = \dim_C H^q(X, \Omega^p)$  and  $P_m(X) = \dim H^0(X, (\Omega^n)^{\otimes m})$  where  $n = \dim_C X$ . Also we denote by  $r$ ,  $\kappa$  and  $b_i$  respectively the number of linearly independent closed holomorphic 1-forms, Kodaira dimension of  $X$  and the  $i$ -th Betti number.

S. Iitaka proposed a problem whether all  $P_m$  and  $\kappa$  are deformation invariants [1]. However computing the numerical characters of small deformations obtained in the above examples we have

**Theorem 2.**  $h^{p,q}$  for  $(p, q) \neq (0, 0)$ ,  $r$ ,  $P_m$  and  $\kappa$  are not necessarily invari-

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ant under small deformations.

On the other hand we note that small deformations of a complex parallelisable manifold are not necessarily parallelisable.

In § 4 and § 5 we prove the following theorems.

**Theorem 3 (Kodaira).** *Let  $X$  be parallelisable such that the corresponding Lie group is nilpotent. Then  $h^{0,1} = r$ .*

**Theorem 4.** *For a complex solvable manifold whose Lie algebra has the Chevalley decomposition (§ 2) we have  $b_1 = 2r$ .*

We remark that a complex solvable manifold has  $\mathbb{C}^n$  as its universal covering.

**Theorem 5.** *If an  $n$ -dimensional complex solvable manifold satisfies the equality  $h^{0,1} = r$ , then any small deformation has  $\mathbb{C}^n$  as its universal covering.*

In Theorem 5 we cannot remove the assumption that  $h^{0,1} = r$ . In fact, in the case of III-(3b) where we have  $h^{0,1} > r$ , there exist small deformations whose universal covering are not analytically homeomorphic to  $\mathbb{C}^3$ .

In § 6, following the algorithm shown in § 1 we classify complex solvable manifolds of four and five dimensions.

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## 1. Preliminaries

Let  $X$  be a compact complex manifold of dimension  $n$ .

**Definition 1.1.**  $X$  is parallelisable if the holomorphic tangent bundle of  $X$  is complex analytically trivial.

This condition is written in the following ways:

- (1)  $\Theta \cong \mathcal{O}^n$ , where  $\Theta$  is the sheaf of germs of holomorphic vector fields, and  $\mathcal{O}$  is the structure sheaf of  $X$ .
- (2) There exist  $n$  holomorphic vector fields  $\theta_1, \dots, \theta_n$  on  $X$  which are linearly independent at every point on  $X$ .
- (3)  $\Omega^1 \cong \mathcal{O}^n$ , where  $\Omega^1$  is the sheaf of germs of holomorphic 1-forms.
- (4) There exist  $n$  holomorphic 1-forms  $\varphi_1, \dots, \varphi_n$  on  $X$  which are linearly independent at every point on  $X$ .

It is obvious that  $\Omega^p \cong \mathcal{O}^{\binom{n}{p}}$ . Hence  $H^0(X, \Omega^p)$  is spanned by  $\{\varphi_{i_1} \wedge \dots \wedge \varphi_{i_p}, 1 \leq i_1 < \dots < i_p \leq n\}$  and  $h^{p,0} = \binom{n}{p}$ .

**Theorem (Wang [6]).** *Let  $X$  be parallelisable. Then there exist a simply connected, connected complex Lie group  $G$  and a discrete subgroup  $\Gamma$  of  $G$  such that  $X \cong G/\Gamma$ .*

*In particular,  $H^0(X, \Theta) \cong \mathfrak{g}$  where  $\mathfrak{g}$  is the Lie algebra of  $G$ .*

**Definition 1.2.** A complex parallelisable manifold  $X$  is solvable (respectively nilpotent) if the corresponding Lie group  $G$  is solvable (respectively nil-

potent).

Let  $G$  be a connected complex Lie group, and  $\Gamma$  one of its discrete subgroups.

**Definition 1.3.**  $\Gamma$  is uniform in  $G$  if  $G/\Gamma$  is compact.

**Theorem (Mostow [4]).** *Let  $G$  be a connected solvable complex Lie group, and  $\Gamma$  a uniform subgroup of  $G$ . Let  $N$  be the connected, maximal nilpotent normal subgroup of  $G$ . Then  $\Gamma \cap N$  and  $\Gamma N/N$  are uniform in  $N$  and  $G/N$  respectively.*

The original form of this theorem is not restricted to the complex case. This theorem means that for any solvable manifold  $X = G/\Gamma$ , there is the decomposition  $\pi: X \rightarrow B$ , where  $B = (G/N)/(\Gamma N/N)$ , and  $(X, \pi, B)$  is a holomorphic fiber bundle with a typical fiber  $F \cong N/\Gamma \cap N$ . We shall call this decomposition the Mostow decomposition of  $X$ . If  $G$  is solvable, the commutator group  $G' = [G, G]$  is nilpotent.  $G'$  is contained in the maximal nilpotent normal subgroup  $N$ , so that  $G/N$  is abelian. Therefore the base space  $B$  is a complex torus.

In an obvious way, we define the pairing

$$H^0(X, \Omega^p) \times H^0(X^p \wedge \Theta) \rightarrow \mathbb{C}, \quad \varphi \times \theta \rightsquigarrow (\varphi, \theta).$$

The exterior differentiation  $d: H^0(X, \Omega^{p-1}) \rightarrow H^0(X, \Omega^p)$  induces an adjoint map  ${}^t d: H^0(X^p \wedge \Theta) \rightarrow H^0(X^{p-1} \wedge \Theta)$ . Then we obtain

- Lemma 1.1.** (1)  $({}^t d)(\theta \wedge \theta') = -[\theta, \theta']$ ,  $\theta, \theta' \in H^0(X, \Theta)$ .  
 (2)  $({}^t d)(\theta \wedge \theta' \wedge \theta'') = -\theta \wedge ({}^t d)(\theta' \wedge \theta'') - \theta' \wedge ({}^t d)(\theta'' \wedge \theta) - \theta'' \wedge ({}^t d)(\theta \wedge \theta')$ ,  $\theta, \theta', \theta'' \in H^0(X, \Theta)$ .

We omit the proof.

(1) of Lemma 1.1 shows

$$(1.1) \quad (d\varphi, \theta \wedge \theta') = -(\varphi, [\theta, \theta'])$$

for  $\theta, \theta' \in H^0(X, \Theta)$ , and  $\varphi \in H^0(X, \Omega^1)$ . (1) and (2) show that  $d^2 = 0$  is equivalent to the Jacobi's identity.

Let  $\mathfrak{g}$  be a solvable Lie algebra defined over  $\mathbb{C}$ . Then by virtue of Lie's theorem we have a  $\mathbb{C}$ -basis of  $\mathfrak{g}$ :  $\varphi_1, \dots, \varphi_n$  ( $n = \dim_{\mathbb{C}} \mathfrak{g}$ ) such that

$$(1.2) \quad d\varphi_\nu = \xi_\nu \wedge \varphi_\nu + \eta_\nu, \quad \nu = 1, \dots, n,$$

where  $\xi_\nu, \eta_\nu$  are represented by  $\varphi_1, \dots, \varphi_{\nu-1}$ . Since  $d^p \varphi_\nu = 0$ , we have  $d\xi_\nu = 0$ , i.e.,  $\xi_\nu$  is a closed holomorphic 1 form. There it follows from (1.3) that  $\xi_\nu = \sum_{\mu=1}^s a_{\nu\mu} \varphi_\mu$  for some constants  $a_{\nu\mu}$ .

**Lemma 1.2.** *Let  $X$  be a compact complex manifold of dimension  $n$ , and  $\varphi$  a holomorphic  $(n-1)$ -form on  $X$ . Then  $d\varphi = 0$ .*

*Proof.* If  $d\varphi \neq 0$ , then  $i^{-n^2} \int_X d\varphi \wedge d\bar{\varphi} > 0$ . On the other hand,

$i^{-n^2} \int_X d\varphi \wedge d\bar{\varphi} = i^{-n^2} \int_X d(\varphi \wedge d\bar{\varphi}) = 0$ , a contradiction.

From Lemma 1.2 we infer readily

**Lemma 1.3.** *Let  $\{\varphi_i\}$  be a basis of  $H^0(X, \Omega^1)$  which satisfies (1.2). Then*

$$(1.3) \quad \sum_{\nu=r+1}^n \xi_\nu = 0.$$

**Proposition 1.4.** *Let  $G$  be a simply connected, connected solvable complex Lie group. Then  $G$  is biholomorphically equivalent to  $C^n$ , where  $n = \dim_C G$ .*

*Proof.* When  $\dim G = 1$ , we can prove the proposition easily. By induction on  $\dim G$  we shall prove the proposition. When  $\dim G \geq 2$ , there exists a connected normal Lie subgroup  $N$  of  $\dim 1$ .  $(G, \pi, G/N)$  is a holomorphic fiber bundle with fiber  $N$ . Calculating homotopy exact sequences of this fiber bundle, we infer readily that  $N$  and  $G/N$  are simply connected, connected and obviously solvable. By the hypothesis of the induction,  $G/N$  and  $N$  are biholomorphically equivalent to  $C^{n-1}$  and  $C$  respectively. From Oka's principle it follows that  $G$  is biholomorphically equivalent to  $C^n$ .

## 2. Classification of three-dimensional complex solvable manifolds and construction of examples

In this section we shall classify three-dimensional complex solvable manifolds, and use an algorithm to classify higher-dimensional complex solvable manifolds. Let  $X = G/\Gamma$  be a three-dimensional solvable manifold, and  $\varphi_1, \varphi_2, \varphi_3$  be a basis of  $H^0(X, \Omega^1)$ , which satisfies (1.2).

By an elementary calculation together with Lemma 1.4, solvable manifolds  $X$  are classified into the following three classes:

$$\text{III-(1): } d\varphi_\lambda = 0, \quad \lambda = 1, 2, 3,$$

$$\text{III-(2): } d\varphi_1 = 0, \quad d\varphi_2 = 0, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2,$$

$$\text{III-(3): } d\varphi_1 = 0, \quad d\varphi_2 = \varphi_1 \wedge \varphi_2, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_3.$$

Dualizing III (1)-(3) by virtue of (1.1) we can determine the structures of the Lie algebra  $\mathfrak{g}$  of  $G$ .

$$\text{III-(1)'}: [\theta_\lambda, \theta_\nu] = 0, \quad \lambda, \nu = 1, 2, 3,$$

$$\text{III-(2)'}: [\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \quad [\theta_\lambda, \theta_\nu] = 0 \text{ otherwise},$$

$$\text{III-(3)'}: [\theta_1, \theta_2] = -[\theta_2, \theta_1] = -\theta_2,$$

$$[\theta_1, \theta_3] = -[\theta_3, \theta_1] = \theta_3, \quad [\theta_2, \theta_3] = 0.$$

Case III-(1). It is well known that  $X$  is a complex torus.

Case III-(2). In view of Proposition 1.4,  $C^3$  is the universal covering of  $X$ .

Let 0 be the origin of  $C^3$ . We set  $\Phi_\nu(z) = \int_0^z \varphi_\nu, \nu = 1, 2$ . Then  $\Phi_\nu$  is a single valued holomorphic function on  $C^3$ , and  $\varphi_\nu = d\Phi_\nu, \nu = 1, 2$ . Thus  $d\varphi_3 = -d\Phi_1 \wedge d\Phi_2$ , i.e.,  $d(\varphi_3 + \Phi_1 d\Phi_2) = 0$ . We set  $\Phi_3(z) = \int_0^z \varphi_3 + \Phi_1 d\Phi_2$ .  $\Phi_3$  is a single valued holomorphic function on  $C^3$ , and  $\varphi_3 = d\Phi_3 - \Phi_1 d\Phi_2$ . For  $g \in \Gamma$ , we set  $z' = z \cdot g$ . Since  $\varphi_\nu$  is  $\Gamma$ -invariant,  $d\Phi_\nu(z') = d\Phi_\nu(z) (\nu = 1, 2)$ . Thus we have  $\Phi_\nu(z') = \Phi_\nu(z) + \omega_\nu(g)$ , where  $\omega_\nu(g)$  is a constant depending only on  $g$ . Since

$$\begin{aligned} \varphi_3(z') &= d\Phi_3(z') - \Phi_1(z')d\Phi_2(z') \\ &= d\Phi_3(z') - (\Phi_1(z) + \omega_1(g))d\Phi_2(z) , \end{aligned}$$

we obtain

$$\Phi_3(z') = \Phi_3(z) + \omega_1(g)\Phi_2(z) + \omega_3(g) ,$$

for some constant  $\omega_3(g)$  depending only on  $g$ . Define a multiplication  $*$  of  $C^3$  by

$$(z_1, z_2, z_3) * (y_1, y_2, y_3) = (z_1 + y_1, z_2 + y_2, z_3 + y_1 z_2 + y_3) .$$

This multiplication  $*$  makes  $C^3$  a nilpotent complex Lie group with the Lie algebra of type III-(2)'. Hence  $G$  is isomorphic to  $(C^3, *)$  as a complex Lie group.

Case III-(3). Set

$$\Phi_1(z) = \int_0^z \varphi_1 , \quad \Phi_2(z) = \int_0^z e^{-\varphi_1} \varphi_2 , \quad \Phi_3(z) = \int_0^z e^{\varphi_1} \varphi_3 .$$

Since  $d\varphi_1 = d(e^{-\varphi_1} \varphi_2) = d(e^{\varphi_1} \varphi_3) = 0$ ,  $\Phi_\nu$  are single valued holomorphic functions on  $C^3$  and we have  $\varphi_1 = d\Phi_1, \varphi_2 = e^{\varphi_1} d\Phi_2, \varphi_3 = d^{-\varphi_1} d\Phi_3$ . By the same argument as in the case of III-(2), we have

$$\begin{aligned} \Phi_1(z') &= \Phi_1(z) + \omega_1(g) , \\ \Phi_2(z') &= e^{-\omega_1(g)} \Phi_2(z) + \omega_2(g) , \\ \Phi_3(z') &= e^{\omega_1(g)} \Phi_3(z) + \omega_3(g) , \end{aligned}$$

where  $z' = z \cdot g$  for  $g \in \Gamma$ , and  $\omega_\nu(g)$ 's are constants depending only on  $g$ . Define a multiplication  $*$  of  $C_3$  by

$$(z_1, z_2, z_3) * (y_1, y_2, y_3) = (z_1 + y_1, e^{-y_1} z_2 + y_2, e^{y_1} z_3 + y_3) .$$

The multiplication  $*$  makes  $C^3$  a solvable complex Lie group with the Lie algebra of type III-(3)', so that  $G$  is isomorphic to  $(C^3, *)$ .

**Examples.** Case III-(2). Set

$$G = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 1 & 1 \end{pmatrix}; z_i \in \mathbb{C} \right\} \cong \mathbb{C}^3,$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix}; \omega_i \in \mathbb{Z} + \mathbb{Z}\sqrt{-1} \right\}.$$

The multiplication is defined by

$$\begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2 + \omega_2 & z_3 + \omega_1 z_2 + \omega_3 \\ 0 & 1 & z_1 + \omega_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$X = G/\Gamma$  is called Iwasawa manifold.

Case III-(3a). We take an algebraic integer  $\alpha$  satisfying the equation  $\alpha^2 + 5\alpha + 7 = 0$ . Let  $E$  be an elliptic curve with fundamental periods  $\{1, \alpha\}$ . Let  $H$  be a group of analytic automorphisms of  $\mathbb{C} \times E \times E$  generated by two automorphisms:

$$\begin{aligned} \sigma_1 &: (z_1, z_2, z_3) \mapsto (z_1 + 2\pi i, z_2, z_3), \\ \sigma_2 &: (z_1, z_2, z_3) \mapsto (z_1 + \beta, (-\alpha - 2)z_2, (\alpha + 3)z_3), \end{aligned}$$

where  $\beta = \log \alpha$ , and  $(z_1, z_2, z_3)$  are global coordinates of  $\mathbb{C} \times E \times E$ .  $H$  acts on  $\mathbb{C} \times E \times E$  properly discontinuously, and its action has no fixed points. The quotient manifold  $X = \mathbb{C} \times E \times E/H$  is a parallelisable manifold of type III-(3) with  $h^{0,1} = \dim_{\mathbb{C}} H^1(X, \mathcal{O}) = 1$ .

Case III-(3b). We take a unimodular matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with trace  $A \geq 3$ . Let  $\alpha$  be an eigenvalue of  $A$ , and  $\beta = \log \alpha > 0$ . Let  $E$  be an elliptic curve with fundamental periods  $\{1, \tau\}$ . Let  $H$  be a group of analytic automorphisms of  $\mathbb{C} \times E \times E$  generated by two automorphisms:

$$\begin{aligned} \sigma_1 &: (z_1, z_2, z_3) \mapsto (z_1 + 2\pi i, z_2, z_3), \\ \sigma_2 &: (z_1, z_2, z_3) \mapsto (z_1 + \beta, az_2 + bz_3, cz_2 + dz_3), \end{aligned}$$

where  $(z_1, z_2, z_3)$  denotes the system of global coordinates of  $\mathbb{C} \times E \times E$ .  $H$  operates on  $\mathbb{C} \times E \times E$  properly discontinuously, and its action has no fixed points. The quotient manifold  $X = \mathbb{C} \times E \times E/H$  is a parallelisable manifold of type III-(3) with  $h^{0,1} = 3$ .

By virtue of Theorems 3 and 4 and the proof of Theorem 4, it can be checked that  $h^{0,1} \neq 2$  for a solvable manifold of type III-(3). Thus we obtain

**Theorem 1.** *Three-dimensional solvable manifolds are classified into the following four classes:*

Lie group	$b_1$	$r$	$h^{0,1}$	Structure (Albanese mapping)
(1) abelian	6	3	3	complex torus
(2) nilpotent	4	2	2	$T^1$ -bundle over $T^2$
(3a) solvable	2	1	1	$T^2$ -bundle over $T^1$
(3b) solvable	2	1	3	$T^2$ -bundle over $T^1$

where  $T^1$  and  $T^2$  denote complex tori of dimensions 1 and 2 respectively.

In this section, we have shown how to determine the structures of  $C^3$  as solvable Lie groups. Proposition 2.2 and the statement below show that this algorithm is valid for higher dimensional cases.

Let  $G$  be a simply connected, connected solvable complex Lie group of dim  $n$ .

**Definition.** A solvable Lie algebra  $\mathfrak{g}$  has the Chevalley decomposition if there exist a commutative subalgebra  $\mathfrak{a}$  and the maximal nilpotent ideal  $\mathfrak{n}$  of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$  (direct sum as vector spaces).

Assume  $\mathfrak{g}$  to have the Chevalley decomposition. Then by definition we can choose a basis  $\{\theta_i\}$  of  $\mathfrak{g}$  such that

$$\begin{aligned}
 [\theta_\lambda, \theta_\nu] &= \sum_{\mu \geq \max(\lambda, \nu)} c'_{\mu\lambda\nu} \theta_\mu, \\
 (2.1) \quad [\theta_\lambda, \theta_\nu] &= 0 \quad (1 \leq \lambda, \nu \leq s), \\
 [\theta_\lambda, \theta_\nu] &= \sum_{\mu > \max(\lambda, \nu)} c'_{\mu\lambda\nu} \theta_\mu \quad (s + 1 \leq \lambda, \nu \leq n),
 \end{aligned}$$

where  $c'_{\mu\lambda\nu} = -c'_{\mu\nu\lambda}$ .

Dualizing (2.1) by (1.1) we conclude that there exists a basis  $\{\varphi_i\}$  of right (or left) invariant 1-forms on  $G$  such that

$$(2.2) \quad d\varphi_\mu = \sum c_{\mu\lambda\nu} \varphi_\lambda \wedge \varphi_\nu,$$

where  $c_{\mu\lambda\nu} = -c_{\mu\nu\lambda}$ ,  $c_{\mu\lambda\nu} = 0$  if “ $1 \leq \lambda, \nu \leq s$ ” or “ $s + 1 \leq \lambda, \nu$  and  $\mu \leq \max(\lambda, \nu)$ ” or “ $\mu < \max(\lambda, \nu)$ ”.

Furthermore we can arrange  $d\varphi_\mu$  in the following order:

$$Q_1: \quad d\varphi_1 = 0, \dots, d\varphi_r = 0, \text{ where } r = \dim H^0(X, d\mathcal{O}),$$

$d\mathcal{O}$  denoting the sheaf of germs of closed holomorphic 1-forms on  $X$ ;

$$\begin{aligned}
 (2.3) \quad Q_l: \quad d\varphi_\lambda &= \text{sum of } \varphi_\nu \wedge \varphi_\mu \text{'s for } (\nu, \mu) \in \bigcup_{\rho=1}^{l-1} Q_{l-\rho} \times Q_\rho \\
 & \hspace{15em} (l = 2, 3, \dots);
 \end{aligned}$$

$$Q = \cup Q_k \text{ and } Q_\infty = \{1, \dots, n\} - Q = \{m+1, \dots, n\};$$

any nontrivial linear combination of  $d\varphi_{m+1}, \dots, d\varphi_n$  cannot be represented by a linear combination of  $d\varphi_1, \dots, d\varphi_m$ .

**Proposition 2.1.** *Assume  $\mathfrak{g}$  to have the Chevalley decomposition, and let  $\{\varphi_\lambda\}$  be a basis of right invariant holomorphic 1-forms on  $G$  which satisfy (2.2). Then there exist holomorphic functions  $\Phi_1, \dots, \Phi_n$  on  $G$  such that*

$$(2.4) \quad \begin{aligned} \varphi_\lambda &= d\Phi_\lambda \quad (1 \leq \lambda \leq r), \\ \varphi_\lambda &= \sum_{\nu=s+1}^{\lambda} F_{\lambda\nu}(\Phi) d\Phi_\nu \quad (r+1 \leq \lambda \leq n), \end{aligned}$$

where  $F_{\lambda\nu}(\Phi) = \sum_a F_{\lambda\nu a}(\Phi) \exp(a_1\Phi_1 + \dots + a_s\Phi_s)$ ,  $F_{\lambda\nu a}(\Phi)$  is a polynomial in  $\Phi_1, \dots, \Phi_{\lambda-1}$ , and  $F_{\lambda\lambda}(\Phi) = \exp(a_1^i\Phi_1 + \dots + a_s^i\Phi_s)$ .

*Proof.* By induction on  $n = \dim G$  we shall prove the proposition, which is obvious for  $n = 1$ . Assume (2.1) to be valid for  $\nu \leq n-1$ . Since  $\xi_n = \sum_{\rho=1}^s a_\rho^n \varphi_\rho$ ,

$$\begin{aligned} & d\left(\exp\left(-\sum_{\rho=1}^s a_\rho^n \Phi_\rho\right)\varphi_n\right) \\ &= -\exp\left(-\sum a_\rho^n \Phi_\rho\right)\xi_n \wedge \varphi_n + \exp\left(-\sum a_\rho^n \Phi_\rho\right)(\xi_n \wedge \varphi_n + \eta_n) \\ &= \sum_{1 \leq \lambda < \nu \leq n-1} F_{\lambda\nu}^*(\Phi) d\Phi_\lambda \wedge d\Phi_\nu. \end{aligned}$$

By the hypothesis of the induction together with (2.1), (2.2) and (2.3) we have

$$\begin{aligned} F_{\lambda\nu}^*(\Phi) &= 0 \quad (1 \leq \lambda, \nu \leq s), \\ F_{\lambda\nu}^*(\Phi) &= \sum_a F_{\lambda\nu a}^* \exp(a_1\Phi_1 + \dots + a_s\Phi_s), \end{aligned}$$

where  $F_{\lambda\nu a}^*(\Phi)$  is a polynomial in  $\Phi_1, \dots, \Phi_{n-2}$ . Take  $G_\nu$  such that

$$\partial G_\nu / \partial \Phi_{n-1} = F_{\nu n-1}^* \quad (\nu > s), \quad \partial G_\nu / \partial \Phi_\nu = F_{\nu n-1}^* \quad (\nu \leq s),$$

$$\begin{aligned} & d\left(\exp\left(-\sum a_\rho^n \Phi_\rho\right)\varphi_n + \sum_{\nu>s} G_\nu d\Phi_\nu - \sum_{\nu \leq s} G_\nu d\Phi_{n-1}\right) \\ &= \sum_{\lambda < \nu} F_{\lambda\nu}^* d\Phi_\lambda \wedge d\Phi_\nu - \sum_{\nu>s} F_{\nu n-1}^* d\Phi_\nu \wedge d\Phi_{n-1} - \sum_{\nu \leq s} F_{\nu n-1}^* d\Phi_\nu \wedge d\Phi_{n-1} \\ &\quad + (\text{terms of } d\Phi_\lambda \wedge d\Phi_\nu, \lambda, \nu \leq n-2) \\ &= \sum_{\lambda < \nu \leq n-2} F_{\lambda\nu}^{**} d\Phi_\lambda \wedge d\Phi_\nu, \quad \text{where } F_{\lambda\nu}^{**} = F_{\lambda\nu}^{**}(\Phi_1, \dots, \Phi_{n-1}). \end{aligned}$$

Since  $0 = d\left(\sum_{\lambda < \nu \leq n-2} F_{\lambda\nu}^{**} d\Phi_\lambda \wedge d\Phi_\nu\right) = \frac{\partial F_{\lambda\nu}^{**}}{\partial \Phi_{n-1}} d\Phi_{n-1} \wedge d\Phi_\lambda \wedge d\Phi_\nu + \dots$ , we



have  $\partial F_{\lambda\nu}^{**} / \partial \bar{\Phi}_{n-1} = 0$ . Hence  $F_{\lambda\nu}^{**} = E_{\lambda\nu}^{**}(\bar{\Phi}_1, \dots, \bar{\Phi}_{n-2})$ . Obviously  $F_{\lambda\nu}^{**} = 0$  ( $1 \leq \lambda, \nu \leq s$ ), etc. Thus we obtain the proposition by induction. q.e.d.

By the same way as stated above we can construct a multiplication  $*$  of  $C^n$ . In order to show that this multiplication defines a Lie group structure of  $C^n$  we have only to check the associative law. We can easily do this by using the fact that  $\varphi_\lambda$  is a right invariant 1-form on  $G$  and the multiplication is written explicitly (see below). Hence the multiplication  $*$  makes  $C^n$  a complex Lie group, and  $(C^n, *)$  is isomorphic to the Lie group  $G$ .

The multiplication  $*$  is defined by

$$(z_1, \dots, z_n) * (y_1, \dots, y_n) = (z_1 + y_1, \dots, z_r + y_r, \dots, \exp(-a_1^r y_1 - \dots - a_s^r y_s) z_\nu + y_\nu + F_\nu(z, y), \dots),$$

where  $F_\nu(z, y)$  is expressed in  $z_1, \dots, z_{\nu-1}, y_1, \dots, y_{\nu-1}$ . Therefore we obtain

**Proposition 2.2.** Assume  $\mathfrak{g}$  to have the Chevalley decomposition. Let  $\{\varphi_\lambda\}$  be a basis of right invariant 1-forms on  $G$  which satisfy (2.3). By an appropriate choice of a system of coordinates  $(z_1, \dots, z_n)$  of  $C^n$ ,  $\{\varphi_\lambda\}$  are represented as follows:

$$(2.5) \quad \varphi_\lambda = \begin{cases} dz_\lambda, & 1 \leq \lambda \leq r, \\ \sum_{\nu=s+1}^{\lambda} F_{\lambda\nu}(z) dz_\nu, & r < \lambda, \end{cases}$$

where  $F_{\lambda\nu}(z) = \sum_a F_{\lambda\nu a}(z) \exp\left(\sum_{\rho=1}^s a_\rho z_\rho\right)$ ,  $F_{\lambda\nu a}$  is a polynomial in  $z_1, \dots, z_{\lambda-1}$ ,

and  $F_{\lambda\lambda} = \exp \sum_{\rho=1}^s a_\rho^{\lambda} z_\rho$ .

Dualizing Proposition 2.3 by means of (1.1), we obtain

**Proposition 2.3.** Let  $\{\theta_\lambda\}$  be a dual basis of right invariant vector fields of  $\{\varphi_\lambda\}$ . Then by the same system of coordinates of  $C^n$  as in Proposition 2.2,  $\{\theta_\lambda\}$  are represented as follows:

$$(2.6) \quad \theta_\lambda = \begin{cases} \partial / \partial z_\lambda, & 1 \leq \lambda \leq r, \\ \sum_{\nu=\lambda}^n G_{\lambda\nu}(z) \partial / \partial z_\nu, & r < \lambda, \end{cases}$$

where  $G_{\lambda\nu}(z) = \sum_a G_{\lambda\nu a}(z) \exp\left(\sum_{\rho=1}^s a_\rho z_\rho\right)$  and  $G_{\lambda\nu a}$  is a polynomial in  $z_1, \dots, z_{\nu-1}$ .

### 3. Construction of Kuranishi families of deformations of three-dimensional complex solvable manifolds

In this section we shall calculate small deformations of three-dimensional

complex solvable manifolds constructed in §2. In these cases we see that several numerical characters, such as  $h^{p,q}$  ( $p, q \neq (0, 0)$ ),  $r$ ,  $P_m$  ( $m \geq 1$ ) are not necessarily invariant under small deformations. Moreover, in the case of III-(3b) there are small deformations whose universal covering are not biholomorphically equivalent to  $C^3$ .

Kodaira first calculated small deformations of Iwasawa manifold. In the first part of this section we shall introduce his result.

Case III-(2) Let  $X = C^3/\Gamma$  be Iwasawa manifold.  $g \in \Gamma$  operates on  $C^3$  as follows :

$$z'_1 = z_1 + \omega_1, \quad z'_2 = z_2 + \omega_2, \quad z'_3 = z_3 + \omega_1 z_2 + \omega_3,$$

where  $g = (\omega_1, \omega_2, \omega_3)$  and  $z' = z \cdot g$ . There exist holomorphic 1-forms  $\varphi_1, \varphi_2, \varphi_3$  which are linearly independent at every point on  $X$  and are given by

$$\varphi_1 = dz_1, \quad \varphi_2 = dz_2, \quad \varphi_3 = dz_3 - z_1 dz_2,$$

so that

$$d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2.$$

On the other hand we have holomorphic vector fields  $\theta_1, \theta_2, \theta_3$  on  $X$  given by

$$\theta_1 = \partial_1, \quad \theta_2 = \partial_2 + z_1 \partial_3, \quad \theta_3 = \partial_3,$$

where  $\partial_i$  denotes  $\partial/\partial z_i$ . It is easily seen that

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \quad [\theta_2, \theta_3] = [\theta_1, \theta_3] = 0.$$

In view of Theorem 2 (§4),  $H^{0,1}(X)$  is spanned by  $\bar{\varphi}_1, \bar{\varphi}_2$ . Since  $\Theta$  is isomorphic to  $\mathcal{O}^3$ ,  $H^{0,1}(X, \Theta)$  is spanned by  $\theta_i \bar{\varphi}_\lambda$ ,  $i = 1, 2, 3$ ,  $\lambda = 1, 2$ .

For vector (0, 1)-forms  $\Psi, \tau$ , we define

$$[\Psi, \tau] = \sum_{\alpha, \beta} (\Psi^\alpha \wedge \partial_\alpha \tau^\beta + \tau^\alpha \wedge \partial^\alpha \Psi^\beta) \partial_\beta,$$

where  $\Psi = \sum \Psi^\alpha \partial_\alpha$  and  $\tau = \sum \tau^\beta \partial_\beta$ . (Cf. [3].) We have

$$[\theta_i \bar{\varphi}_\lambda, \theta_k \bar{\varphi}_\nu] = [\theta_i, \theta_k] \bar{\varphi}_\nu \wedge \bar{\varphi}_\nu.$$

We shall construct a vector (0, 1)-form  $\psi$  such that

$$(3.1) \quad \bar{\partial}\psi - \frac{1}{2}[\psi, \psi] = 0.$$

Set  $\psi = \sum_{\alpha=1}^{\infty} \psi_\alpha(t)$ , where  $\psi_1(t) = \sum_{i=1}^3 \sum_{\lambda=1}^2 t_{i\lambda} \theta_i \bar{\varphi}_\lambda$ , and  $\psi_\alpha(t)$  is the homogeneous term of total degree  $\alpha$  in  $t_{i\lambda}$ . Then

$$\bar{\partial}\psi_2(t) = \frac{1}{2}[\psi_1(t), \psi_1(t)] = (t_{11}t_{22} - t_{21}t_{12})\theta_3\bar{\varphi}_1 \wedge \bar{\varphi}_2 .$$

Set  $\psi_2(t) = -(t_{11}t_{22} - t_{21}t_{12})\theta_3\bar{\varphi}_3$ . Thus we obtain a solution of (3.1) given by

$$\psi(t) = \sum_{i=1}^3 \sum_{\lambda=1}^2 t_{i\lambda}\theta_i\bar{\varphi}_\lambda - (t_{11}t_{22} - t_{21}t_{12})\theta_3\bar{\varphi}_3 .$$

This proves the existence of a locally complete complex analytic family of deformations  $X_t$  of  $X$  depending on 6 effective parameters  $t_{i\lambda}$ , [3].

Next, by solving the system of differential equations

$$(3.2) \quad \bar{\partial}\zeta_\nu - \psi(t)\zeta_\nu = 0, \quad \nu = 1, 2, 3,$$

under the initial condition  $\zeta_\nu(0) = z_\nu$ , we have the solutions:

$$\begin{aligned} \zeta_1 &= z_1 + \sum_{\lambda=1}^2 t_{1\lambda}\bar{z}_\lambda, & \zeta_2 &= z_2 + \sum_{\lambda=1}^2 t_{2\lambda}\bar{z}_\lambda, \\ \zeta_3 &= z_3 + \sum_{\lambda=1}^2 (t_{3\lambda} + t_{2\lambda}z_1)\bar{z}_\lambda + A(\bar{z}) - D(t)\bar{z}_3, \end{aligned}$$

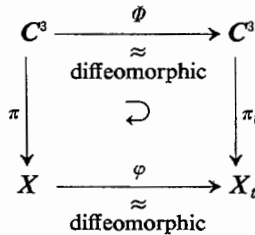
where

$$D(t) = t_{11}t_{22} - t_{21}t_{12}, \quad A(\bar{z}) = \frac{1}{2}(t_{11}t_{21}\bar{z}_1^2 + 2t_{11}t_{22}\bar{z}_1\bar{z}_2 + t_{12}t_{22}\bar{z}_2^2) .$$

Since

$$\begin{aligned} d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\bar{\zeta}_3 \\ = c(t)dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3, \end{aligned}$$

where  $c(t)$  is a differentiable function in  $t_{i\lambda}$  with  $c(0) = 1$ , it follows that  $\Phi: (z_1, z_2, z_3) \mapsto (\zeta_1, \zeta_2, \zeta_3)$  is a diffeomorphism of  $\mathbb{C}^3$  if  $\sum_{i,\lambda} |t_{i\lambda}| < \varepsilon$  for sufficiently small positive number  $\varepsilon$ .



Since  $\pi$  is a covering map,  $\pi_t = \varphi \circ \pi \circ \Phi^{-1}$  is also a covering map from  $\mathbb{C}^3$  to  $X_t$ . Therefore  $\mathbb{C}^3$  is the universal covering of  $X_t$ , that is,  $X_t = \mathbb{C}^3 / \Gamma_t$  for a group  $\Gamma_t$  of analytic automorphisms of  $\mathbb{C}^3$ . The group  $\Gamma_t$  is defined by

$$\zeta'_1 = \zeta_1 + \tilde{\omega}_1(t), \quad \zeta'_2 = \zeta_2 + \tilde{\omega}_2(t),$$

$$\zeta'_3 = \zeta_3 + \tilde{\omega}_3(t) + \omega_1 \zeta_2 + \left( \sum_{\lambda=1}^2 t_{2\lambda} \bar{\omega}_\lambda \right) \zeta_1 + A(\bar{\omega}) - D(t) \bar{\omega}_3,$$

where  $\tilde{\omega}_i(t) = \omega_i + t_{i1} \bar{\omega}_1 + t_{i2} \bar{\omega}_2$  for  $(\omega_1, \omega_2, \omega_3) \in \Gamma$ .

Now we summarize the numerical characters of deformations. *The deformations are divided into the following three classes:*

- i)  $t_{11} = t_{12} = t_{21} = t_{22} = 0$ ,  $X_t$  is a parallelisable manifold of type III-(2).
- ii)  $D(t) = 0$  and  $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$ ,  $X_t$  is not parallelisable.
- iii)  $D(t) \neq 0$ ,  $X_t$  is not parallelisable.

	$r$	$h^{1,0}$	$h^{0,1}$	$h^{2,0}$	$h^{1,1}$	$h^{0,2}$	$h^{3,0}$	$h^{2,1}$	$h^{1,2}$	$h^{0,3}$	$P_m (m \geq 1)$
i)	2	3	2	3	6	2	1	6	6	1	1
ii)	2	2	2	2	5	2	1	5	5	1	1
iii)	2	2	2	1	5	2	1	4	4	1	1

$$h^{3-p,3-q} = h^{p,q}$$

Next we shall calculate small deformations of a solvable manifold of type III-(3) constructed in § 2. As stated before,  $h^{0,1}(X) = 1$  or 3 (see the proof of Theorem 3).

First we shall consider the case where  $h^{0,1} = 3$ . Let  $X = \mathcal{C}^3/\Gamma$  be a solvable manifold constructed in Example III-(3b). By an appropriate linear transformation of  $z_2$  and  $z_3$ ,  $g \in \Gamma$  operates on  $\mathcal{C}^3$  as follows:

$$z'_1 = z_1 + \omega_1, \quad z'_2 = e^{-\omega_1} z_2 + \omega_2, \quad z'_3 = e^{\omega_1} z_3 + \omega_3.$$

There exist holomorphic 1-forms  $\varphi_1, \varphi_2, \varphi_3$  on  $X$  given by

$$\varphi_1 = dz_1, \quad \varphi_2 = e^{z_1} dz_2, \quad \varphi_3 = e^{-z_1} dz_3,$$

so that

$$d\varphi_1 = 0, \quad d\varphi_2 = \varphi_1 \wedge \varphi_2, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_3.$$

On the other hand, there exist holomorphic vector fields  $\theta_1, \theta_2, \theta_3$  given by

$$\theta_1 = \partial_1, \quad \theta_2 = e^{-z_1} \partial_2, \quad \theta_3 = e^{z_1} \partial_3,$$

such that

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = -\theta_2, \quad [\theta_1, \theta_3] = -[\theta_3, \theta_1] = \theta_3, \quad [\theta_2, \theta_3] = 0.$$

$H^{0,1}_\theta(X)$  is generated by  $\varphi_1^* = d\bar{z}_1$ ,  $\varphi_2^* = e^{z_1} d\bar{z}_2$  and  $\varphi_3^* = e^{-z_1} d\bar{z}_3$  (see the proof of Theorem 3). Since  $\varphi_2^* = e^{z_1 - z_2} \bar{\varphi}_2$  and  $\varphi_3^* = e^{-z_1 + z_3} \bar{\varphi}_3$ ,  $H^{0,1}_\theta(X, \Theta)$  is spanned by  $\theta_i \varphi_i^*$ ,  $i = 1, 2, 3$ ,  $\lambda = 1, 2, 3$ . We shall construct a vector (0, 1)-forms  $\psi$  satisfying (3.1).

Set  $\psi(t) = \sum_{\alpha=1}^{\infty} \psi_{\alpha}(t)$ , where  $\psi_1(t) = \sum_{i=1}^3 \sum_{\lambda=1}^3 t_{i\lambda} \theta_i \varphi_{\lambda}^*$  and  $\psi_{\alpha}(t)$  is the homogeneous term of total degree  $\alpha$  in  $t_{i\lambda}$ . Then we have

$$\begin{aligned} [\theta_i \varphi_1^*, \theta_{\mu} \varphi_3^*] &= (\delta_{i\lambda} \theta_{\mu} + [\theta_{\lambda}, \theta_{\mu}]) \varphi_1^* \wedge \varphi_3^*, \\ [\theta_{\lambda} \varphi_1^*, \theta_{\mu} \varphi_3^*] &= (-\delta_{i\lambda} \theta_{\mu} + [\theta_{\lambda}, \theta_{\mu}]) \varphi_1^* \wedge \varphi_3^*, \\ [\theta_i \varphi_2^*, \theta_{\mu} \varphi_3^*] &= (-\delta_{i\lambda} \theta_{\mu} - \delta_{1\mu} \theta_{\lambda} + [\theta_{\lambda}, \theta_{\mu}]) \varphi_2^* \wedge \varphi_3^*, \\ \frac{1}{2}[\psi_1, \psi_1] &= \eta_1 \theta_1 + \eta_2 \theta_2 + \eta_3 \theta_3, \end{aligned}$$

where

$$\begin{aligned} \eta_1 &= -t_{11} t_{13} \varphi_1^* \wedge \varphi_3^* + t_{11} t_{12} \varphi_1^* \wedge \varphi_2^* - 2t_{12} t_{13} \varphi_2^* \wedge \varphi_3^*, \\ \eta_2 &= (t_{21} t_{13} - 2t_{11} t_{23}) \varphi_1^* \wedge \varphi_3^* + t_{12} t_{21} \varphi_1^* \wedge \varphi_2^* - 2t_{12} t_{23} \varphi_2^* \wedge \varphi_3^*, \\ \eta_3 &= -t_{13} t_{31} \varphi_1^* \wedge \varphi_3^* + (2t_{11} t_{32} - t_{31} t_{12}) \varphi_1^* \wedge \varphi_2^* - 2t_{13} t_{32} \varphi_2^* \wedge \varphi_3^*. \end{aligned}$$

Since  $\bar{\partial} \psi_2 = \frac{1}{2}[\psi_1, \psi_1]$ , it follows that  $\eta_{\nu}$  is cohomologous to zero in  $H_{\mathbb{R}}^{0,2}(X)$ .

**Lemma 3.1.** *Set  $\eta = A \varphi_1^* \wedge \varphi_3^* + B \varphi_1^* \wedge \varphi_2^* + C \varphi_2^* \wedge \varphi_3^*$ , and assume that  $\eta$  is cohomologous to zero in  $H_{\mathbb{R}}^{0,2}(X)$ . Then  $A = B = C = 0$ .*

*Proof.*  $\varphi_1^* \wedge \varphi_3^* = e^{z_1 - \bar{z}_1} \bar{\varphi}_1 \wedge \bar{\varphi}_3$ ,  $\varphi_2^* \wedge \varphi_3^* = \bar{\varphi}_2 \wedge \bar{\varphi}_3$ ,  $\varphi_1^* \wedge \varphi_2^* = e^{z_1 - \bar{z}_1} \bar{\varphi}_1 \wedge \bar{\varphi}_2$ ,  $\bar{\partial}(\varphi_i^* \wedge \varphi_{\nu}^*) = 0$ . If  $f_1, f_2, f_3$  are functions in  $z_1, \bar{z}_1$ , then

$$\mathcal{D}(f_1 \bar{\varphi}_2 \wedge \bar{\varphi}_3 + f_2 \bar{\varphi}_1 \wedge \bar{\varphi}_3 + f_3 \bar{\varphi}_1 \wedge \bar{\varphi}_2) = -(\partial_1 f_2 + f_2) \bar{\varphi}_3 - (\partial_1 f_3 - f_3) \bar{\varphi}_2,$$

where  $\mathcal{D}$  is the adjoint operator of  $\bar{\partial}$  (see the proofs of Theorems 2 and 3). Thus  $\mathcal{D}(\varphi_1^* \wedge \varphi_3^*) = \mathcal{D}(\varphi_1^* \wedge \varphi_2^*) = \mathcal{D}(\varphi_2^* \wedge \varphi_3^*) = 0$ , and  $\varphi_i^* \wedge \varphi_{\nu}^*$  is harmonic. Hence  $\eta = 0$ ,  $A = B = C = 0$ . q.e.d.

It follows from Lemma 3.1 that

$$(3.3) \quad \begin{aligned} t_{11} t_{13} &= 0, & t_{11} t_{12} &= 0, & t_{12} t_{13} &= 0, \\ t_{21} t_{13} - 2t_{11} t_{23} &= 0, & t_{12} t_{21} &= 0, & t_{12} t_{23} &= 0, \\ t_{31} t_{13} &= 0, & 2t_{11} t_{32} - t_{31} t_{12} &= 0, & t_{13} t_{32} &= 0. \end{aligned}$$

Consequently  $\psi = \psi_1$ .

By solving (3.2) we have the solutions:

$$\begin{aligned} \eta_1 &= z_1 + t_{11} \bar{z}_1 - \log(1 - t_{12} e^{z_1 \bar{z}_2}) + \log(1 + t_{13} e^{-z_1 \bar{z}_3}), \\ \eta_2 &= z_2 + t_{22} \bar{z}_2 - t_{21} t_{13} e^{-2z_1} \bar{z}_1 \bar{z}_3 - \frac{t_{21}}{t_{11}} e^{z_1} (e^{-t_{11} \bar{z}_1} - 1) + \frac{t_{23} e^{-2z_1} \bar{z}_3}{1 - t_{13} e^{-z_1 \bar{z}_3}}, \\ \eta_3 &= z_3 + t_{33} \bar{z}_3 + t_{12} t_{31} e^{2z_1} \bar{z}_1 \bar{z}_2 + \frac{t_{31}}{t_{11}} e^{z_1} (e^{t_{11} \bar{z}_1} - 1) + \frac{t_{32} e^{2z_1} \bar{z}_2}{1 - t_{12} e^{z_1 \bar{z}_2}}. \end{aligned}$$

Four cases may occur. If  $t_{12} = t_{13} = 0$ , we infer that  $C^3$  is the universal covering of  $X_t$  by the same argument as in the case of III-(2).

$$\begin{aligned} \text{Case 1: } \quad & t_{11} \neq 0, \quad t_{12} = t_{13} = t_{23} = t_{32} = 0, \quad \zeta_1 = z_1 + t_{11}\bar{z}_1, \\ & \zeta_2 = z_2 + t_{22}\bar{z}_2 - \frac{t_{21}}{t_{11}} e^{-z_1}(e^{-t_{11}z_1} - 1), \\ & \zeta_3 = z_3 + t_{33}\bar{z}_3 + \frac{t_{31}}{t_{11}} e^{z_1}(e^{t_{11}z_1} - 1). \end{aligned}$$

$\mathcal{C}^3$  is the universal covering of  $X_t$ , i.e.,  $X_t = \mathcal{C}^3/\Gamma_t$  for a group  $\Gamma_t$  of analytic automorphisms of  $\mathcal{C}^3$ ; the group  $\Gamma_t$  is defined by

$$\begin{aligned} \zeta'_1 &= \zeta_1 + \tilde{\omega}_1, \\ \zeta'_2 &= e^{-\omega_1}\zeta_2 + \tilde{\omega}_2 + \frac{t_{21}}{t_{11}} e^{-\zeta_1 - \omega_1}(1 - e^{-t_{11}\tilde{\omega}_1}), \\ \zeta'_3 &= e^{\omega_1}\zeta_3 + \tilde{\omega}_3 - \frac{t_{31}}{t_{11}} e^{\zeta_1 + \omega_1}(1 - e^{t_{11}\tilde{\omega}_1}), \end{aligned}$$

where  $\tilde{\omega}_i = \omega_i + t_{ii}\bar{\omega}_i$  for  $(\omega_1, \omega_2, \omega_3) \in \Gamma$ .

$$\begin{aligned} \text{Case 2: } \quad & t_{11} = t_{12} = t_{13} = 0, \quad \zeta_1 = z_1, \\ & \zeta_2 = z_2 + t_{22}\bar{z}_2 + t_{21}e^{-z_1}\bar{z}_1 + t_{23}e^{-2z_1}\bar{z}_3, \\ & \zeta_3 = z_3 + t_{33}\bar{z}_3 + t_{31}e^{z_1}\bar{z}_1 + t_{32}e^{2z_1}\bar{z}_2. \end{aligned}$$

$\mathcal{C}^3$  is also the universal covering of  $X_t$ , i.e.,  $X_t = \mathcal{C}^3/\Gamma_t$ ;  $\Gamma_t$  is defined by

$$\begin{aligned} \zeta'_1 &= \zeta_1 + \omega_1, \\ \zeta'_2 &= e^{-\omega_1}\zeta_2 + \tilde{\omega}_2 + t_{21}\bar{\omega}_1 e^{-\zeta_1 - \omega_1} + t_{23}\bar{\omega}_3 e^{-2\zeta_1 - 2\omega_1}, \\ \zeta'_3 &= e^{\omega_1}\zeta_3 + \tilde{\omega}_3 + t_{31}\bar{\omega}_1 e^{\zeta_1 + \omega_1} + t_{32}\bar{\omega}_2 e^{2\zeta_1 + 2\omega_1}, \end{aligned}$$

where  $\tilde{\omega}_i = \omega_i + t_{ii}\bar{\omega}_i$ ,  $i = 2, 3$  for  $(\omega_1, \omega_2, \omega_3) \in \Gamma$ .

$$\begin{aligned} \text{Case 3 (Kodaira): } \quad & t_{12} \neq 0, \quad t_{11} = t_{13} = t_{21} = t_{23} = t_{31} = 0, \\ & \zeta_1 = z_1 - \log(1 - t_{12}e^{z_1}\bar{z}_2), \quad \zeta_2 = z_2 + t_{22}\bar{z}_2, \\ & \zeta_3 = z_3 + t_{33}\bar{z}_3 + \frac{t_{32}e^{2z_1}\bar{z}_2}{1 - t_{12}e^{z_1}\bar{z}_2}. \end{aligned}$$

Set

$$\begin{aligned} w &= e^{-z_1}, \quad \eta_1 = w - t_{12}\bar{z}_2, \quad \eta_2 = z_2 + t_{22}\bar{z}_2, \\ \eta_3 &= z_3 + t_{33}\bar{z}_3 - \frac{t_{32}}{t_{12}} \frac{1}{w}. \end{aligned}$$

Any  $g \in \Gamma$  induces a transformation  $g_t$  of  $W_t$  as follows:

$$\eta'_1 = e^{-\omega_1}(\eta_1 - t_{12}\bar{\omega}_2), \quad \eta'_2 = e^{-\omega_1}(\eta_2 + \bar{\omega}_2), \quad \eta'_3 = e^{\omega_1}(\eta_3 + \bar{\omega}_3),$$

where  $W_t = \{(\eta_1, \eta_2, \eta_3) \in \mathbb{C}^3; (1 - |t_{22}|^2)\eta_1 + t_{12}(\bar{\eta}_2 - \bar{t}_{22}\eta_2) \neq 0\}$  and  $\bar{\omega}_i = \omega_i + t_{ii}\bar{\omega}_i, i = 2, 3$  for  $(\omega_1, \omega_2, \omega_3) \in \Gamma$ . Set  $\Delta_t = \{g_t; g \in \Gamma\}$ . Then we have  $X_t = W_t/\Delta_t$  for  $\sum_{i,j} |t_{ij}| < 1$  and  $X = X_0 = W_0/\Delta_0$ . For  $t_{12} \neq 0, W_t$  is not a domain of holomorphy. In fact, by virtue of the edge of the wedge theorem [6] any multivalued holomorphic function on  $W_t$  extends to  $\mathbb{C}^3$ . In particular the universal covering manifold  $\tilde{W}_t$  of  $W_t$  cannot be imbedded into  $\mathbb{C}^n$  for any  $n$ .

Case 4:  $t_{12} \neq 0, \quad t_{11} = t_{12} = t_{21} = t_{31} = t_{32} = 0$ .

By the transformation:  $(z_1, z_2, z_3) \mapsto (-z_1, -z_2, -z_3), (\zeta_1, \zeta_2, \zeta_3) \mapsto (-\zeta_1, -\zeta_2, -\zeta_3)$ , we can reduce Case 4 to Case 3.

Now we summarize the numerical characters of small deformations in Case 3.

	$r$	$h^{10}$	$h^{01}$	$h^{02}$	$h^{30}$	$h^{03}$	$h^{31}$	$h^{32}$	$P_m(m \geq 1)$	$\kappa$
i) $t_{\lambda i} = 0$ $i, \lambda = 1, 2, 3$	1	3	3	3	1	1	3	3	1	0
ii) $t_{12} \neq 0$	0	0	2	1	0	0	2	1	0	$-\infty$

Thus we obtain

**Theorem 2.**  $h^{p,q} (p, q) \neq (0, 0), r, P_m (m \geq 1)$  and  $\kappa$  are not necessarily invariant under small deformations.

Secondly we shall consider the case where  $X$  is of type III-(3) with  $h^{0,1} = 1$ . Holomorphic 1-forms and vector fields on  $X$  are given as follows:

$$\begin{aligned} \varphi_1 &= dz_1, \quad \varphi_2 = e^{z_1} dz_2, \quad \varphi_3 = e^{-z_1} dz_3, \\ \theta_1 &= \partial_1, \quad \theta_2 = e^{-z_1} \partial_2, \quad \theta_3 = e^{z_1} \partial_3. \end{aligned}$$

$H^{0,1}(\theta)$  is spanned by  $\theta_1\bar{\varphi}_1, \theta_2\bar{\varphi}_1, \theta_3\bar{\varphi}_1$ , and the vector (0, 1)-form  $\psi$  satisfying (3.1) is given by  $\psi(t) = \sum_{i=1}^3 t_i \theta_i \bar{\varphi}_1$ . We can construct a locally complete complex analytic family of deformations of  $X$  depending on 3 effective parameters  $t_i$ .

Case 1:  $t_1 \neq 0, \quad \zeta_1 = z_1 + t_1\bar{z}_1,$

$$\zeta_2 = z_2 - \frac{t_2}{t_1} e^{-z_1} (e^{-t_1 z_1} - 1), \quad \zeta_3 = z_3 + \frac{t_3}{t_1} e^{z_1} (e^{t_1 z_1} - 1).$$

$X^t = \mathbb{C}^3/\Gamma_t$ , and the group  $\Gamma_t$  is defined by

$$\zeta'_1 = \zeta_1 + \omega_1 + t_1\bar{\omega}_1, \quad \zeta'_2 = e^{-\omega_1}\zeta_2 + \frac{t_2}{t_1} e^{-\zeta_1 - \omega_1} (1 - e^{-t_1\bar{\omega}_1}),$$

$$\zeta'_3 = e^{\omega_1} \zeta_3 - \frac{t_3}{t_1} e^{\zeta_1 + \omega_1} (1 - e^{t_1 \bar{\omega}_1}) \quad \text{for } (\omega_1, \omega_2, \omega_3) \in \Gamma .$$

Case 2:  $t_1 = 0$  ,  $\zeta_1 = z_1$  ,  $\zeta_2 = z_2 + t_2 e^{-z_1 \bar{z}_1}$  ,  $\zeta_3 = z_3 + t_3 e^{z_1 \bar{z}_1}$  .

$x^t = \mathcal{C}^3 / \Gamma_t$ , and the group  $\Gamma_t$  is defined by

$$\begin{aligned} \zeta'_1 &= \zeta_1 + \omega_1 , & \zeta'_2 &= e^{-\omega_1} \zeta_2 + \omega_2 + t_2 \bar{\omega}_1 e^{-\zeta_1 - \omega_1} , \\ \zeta'_3 &= e^{\omega_1} \zeta_3 + \omega_3 + t_3 \bar{\omega}_1 e^{\zeta_1 + \omega_1} & \text{for } (\omega_1, \omega_2, \omega_3) \in \Gamma . \end{aligned}$$

#### 4. Proofs of Theorems 2 and 3

The following theorem is due to Kodaira.

**Theorem 3.** *If  $X$  is nilpotent, then  $h^{01} = r$ .*

*Proof.* We shall calculate the dimension of harmonic  $(0, 1)$ -forms by the Dolbeault isomorphism  $H^1(X, \mathcal{O}) \cong H_{\bar{\partial}}^{01}(X)$ . Let  $\{\varphi_i\}$  and  $\{\theta_i\}$  be a basis of  $H^0(X, \Omega^1)$  and  $H^0(X, \Theta)$  dual to each other with respect to (1.1), which satisfy (2.1) and (2.2) respectively. Let  $\varphi$  be a differentiable  $(0, 1)$ -form on  $X$ . Then  $\varphi = \sum_{i=1}^n f_i \bar{\varphi}_i$ , where  $f_i$ 's are differentiable function on  $X$ , so that

$$\bar{\partial}\varphi = \sum_{i, \nu} (\bar{\theta}_\nu f_\nu) \bar{\varphi}_\nu \wedge \bar{\varphi}_i + \sum_{i=1}^n f_i d\bar{\varphi}_i = \sum_{\nu < i} (\bar{\theta}_i f_\nu - \bar{\theta}_\nu f_i + 2 \sum_{\mu=1}^n \overline{c_{\mu\nu i}} f_\mu) \bar{\varphi}_\nu \wedge \bar{\varphi}_i .$$

For a differentiable  $(0, 1)$ -form  $\gamma = \sum_{i=1}^n g_i \bar{\varphi}_i$  we define

$$(\varphi, \lambda) = \int_X \sum_{i=1}^n f_i \bar{g}_i dx ,$$

where  $dX = i^{-n^2} \varphi_1 \wedge \dots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \dots \wedge \bar{\varphi}_n$ .

Let  $\mathcal{G}$  be the adjoint operator or  $\bar{\partial}$  with respect to the inner  $( , )$ . For a differentiable function  $g$  we have

$$(\mathcal{G}\varphi, g) = (\varphi, \bar{\partial}g) = \int_X \sum_{i=1}^n f_i \theta_i \bar{g} dX = - \int_X (\sum \theta_i f_i) \bar{g} dX .$$

Hence  $\mathcal{G}\varphi = - \sum_{i=1}^n \theta_i f_i$ . Assume that  $\varphi$  is harmonic so that  $\bar{\partial}\varphi = 0$ ,  $\mathcal{G}\varphi = 0$ . Consequently

$$\bar{\theta}_\nu f_i - \bar{\theta}_i f_\nu + 2 \sum_{\mu=1}^n \overline{c_{\mu\nu i}} f_\mu = 0, \quad \sum_{i=1}^n \theta_i f_i = 0 .$$

Define the Laplacian  $\square$  by

$$\square = \mathcal{G}\bar{\partial} + \bar{\partial}\mathcal{G} .$$



Then  $\square f = 0$  implies  $\bar{\partial}f = 0$  for a function  $f$ . Hence  $f$  is holomorphic on  $X$ , and is constant.

$$\square f_\nu = -2 \sum_{\lambda, \mu=1}^n \overline{c_{\nu\lambda}} \theta_\lambda f_\mu .$$

Since  $X$  is nilpotent, we have  $c_{\nu\lambda} = 0$  ( $\nu \geq \mu$  or  $\lambda \geq \mu$ ). Thus  $\square f_n = 0$ , which implies that  $f_n$  is constant. From this it follows that

$$\square f_{n-1} = -2 \sum_{\lambda=1}^n \overline{c_{n-1\lambda}} \theta_\lambda f_n = 0 .$$

Thus  $f_{n-1}$  is constant. Inductively we conclude that any  $f_\nu$  is constant. Since  $\bar{\partial} = 0$ ,  $f_\lambda = 0$  ( $r < \lambda$ ). Hence  $\varphi = \sum_{\lambda=1}^n f_\lambda \bar{\varphi}_\lambda$  where  $f_\lambda$ 's are constant, i.e.,  $h^{0,1} = r$ .

**Theorem 4.** *If  $X$  is solvable and its Lie algebra has the Chevalley decomposition, then we have  $b_1 = 2r$ .*

*Proof.* First we assume  $X$  to be nilpotent. Consider the following exact sequence:

$$0 \longrightarrow C \longrightarrow \mathcal{O} \xrightarrow{d} d\mathcal{O} \longrightarrow 0 .$$

Then we have

$$0 \rightarrow H^0(X, d\mathcal{O}) \rightarrow H^1(X, C) \rightarrow H^1(X, \mathcal{O}) \rightarrow \dots .$$

From theorem 3 it follows that  $b_1 \leq 2r$ , while in general  $b_1 \geq 2 \dim_C H^0(X, d\mathcal{O})$ . Hence we complete the proof in case that  $X$  is nilpotent.

Now we assume  $X$  not to be nilpotent. Then the Mostow decomposition  $(X, \pi, B)$  is nontrivial. Set  $\dim B = s$  ( $\geq 1$ ) and  $\dim X = n$ . Then we can take a system of coordinates  $(z_1, \dots, z_n)$  of the universal covering  $C^n$  of  $X$  satisfying the following two conditions:

(1)  $\pi$  is the projection to the first  $s$  factors, and  $(X, \pi, B)$  is a holomorphic fiber bundle with nilpotent  $F$  as fiber:

$$\begin{array}{ccc} X = C^n / \Gamma \ni (z_1, \dots, z_n) & & \\ \pi \downarrow & & \downarrow \\ B = C^s / \Gamma \ni (z_1, \dots, z_s) . & & \end{array}$$

(2)  $\theta_\lambda, \varphi_\nu$  are represented in the forms (2.5) and (2.6), and  $g_1 \in \Gamma_1$  induces an analytic automorphism of  $F$  and  $\text{Alb } F$ ; hence  $g_1$  operates on  $(z_{s+1}, \dots, z_{s+r(F)})$  as an affine transformation.

Denoting the  $\nu$ -th coordinate of  $z \cdot g_1$  by  $(z \cdot g_1)_\nu = z'_\nu$ , we have  $z'_\nu = \sum_{\mu=s+1}^n a_{\nu\mu} z_\mu + c_\nu$ ,  $s+1 \leq \nu \leq s+r(F)$ , where  $a_{\nu\mu}$  is constant and  $c_\nu = c_\nu(z_1, \dots, z_s, g_1)$ . By induction on  $\nu$  we can check that any  $c_\nu$  is constant depending only on  $g_1$

in view of the representation (2.5) of  $\varphi_\nu$ . Consider the following spectral sequence:

$$E_2^{p,q} = H^p(B, R^q\pi_*\mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}).$$

Then we have the exact sequence:

$$0 \rightarrow H^1(B, \mathcal{O}_B) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^0(B, R^1\pi_*\mathcal{O}_X) \rightarrow \dots$$

$$\parallel$$

$$H_3^{0,1}(X)$$

Since  $F$  is nilpotent,  $H_3^{0,1}(F, \mathcal{O}_F)$  is generated by  $d\bar{z}_{s+1}, \dots, d\bar{z}_{s+r(F)}$ , and therefore any element  $\psi$  of  $H^0(B, R^1\pi_*\mathcal{O}_X)$  can be written in the form  $\psi = \sum_{\lambda=s+1}^{s+r(F)} f_\lambda(z) d\bar{z}_\lambda$  where  $f_\lambda(z)$  is holomorphic in  $z_1, \dots, z_s$ . By the above arguments,  $\psi$  can be viewed as a  $(0, 1)$ -form on  $X$ , and can therefore be written as  $\psi = \sum_{\lambda=s+1}^n g_\lambda \bar{\varphi}_\lambda$  where  $g_\lambda = g_\lambda(z_1, \dots, z_s, \bar{z}_1, \dots, \bar{z}_n)$  is antiholomorphic in  $z_{s+1}, \dots, z_n$ .

By Proposition 2.3 we see readily that  $\bar{\partial}\psi = 0$ ,  $\mathcal{G}\psi = -\sum_{\lambda=s+1}^n \theta_\lambda g_\lambda = 0$ , and consequently that  $\psi$  itself can be viewed as an element of  $H^1(X, \mathcal{O}) \cong H_3^{0,1}(X)$ . Hence  $H^1(X, \mathcal{O}_X) = H^1(B, \mathcal{O}_B) \oplus H^0(B, R^1\pi_*\mathcal{O}_X)$ , that is, any element  $\psi$  of  $H_3^{0,1}(X)$  can be represented in the form

$$\psi = \sum_{\lambda=1}^s c_\lambda d\bar{z}_\lambda + \sum_{\lambda=s+1}^{s+r(F)} f_\lambda(z) d\bar{z}_\lambda = \sum_{\lambda=1}^s c_\lambda d\bar{z}_\lambda + \sum_{\lambda=s+1}^n g_\lambda \bar{\varphi}_\lambda,$$

where  $c_\lambda$  is constant, and  $f_\lambda, g_\lambda$  are the same as above. We shall calculate the dimension of real harmonic 1-forms on  $X$ . Let  $\varphi$  be a real differentiable 1-form given by

$$\varphi = \sum_{\lambda=1}^n \bar{g}_\lambda \varphi_\lambda + \sum_{\lambda=1}^n g_\lambda \bar{\varphi}_\lambda,$$

where  $g_\lambda$  is a differentiable function. Set  $\psi = \sum_{\lambda=1}^n g_\lambda \bar{\varphi}_\lambda$ . Then  $\varphi = \bar{\psi} + \psi$ . Define  $d, \delta$  by

$$d\varphi = (\partial + \bar{\partial})(\psi + \bar{\psi}), \quad \delta\varphi = (\mathcal{G} + \bar{\mathcal{G}})(\psi + \bar{\psi}).$$

Assume  $\varphi$  is harmonic. Then  $d\varphi = 0, \delta\varphi = 0$ , and therefore  $\bar{\partial}\psi = 0, \partial\bar{\psi} + \partial\psi = 0$ . Since  $\partial\bar{\psi} + \partial\psi = \sum (\theta_\nu g_\lambda - \bar{\theta}_\lambda \bar{g}_\nu) \varphi_\lambda \wedge \bar{\varphi}_\nu$ , we have

$$(4.1) \quad \theta_\nu g_\lambda = \overline{\theta_\lambda g_\nu}.$$

On the other hand,

$$0 = \bar{\partial}\varphi = -\sum_{\lambda=1}^n \overline{\theta_\lambda g_\lambda} - \sum_{\lambda=1}^n \theta_\lambda g_\lambda = -2 \sum_{\lambda=1}^n \theta_\lambda g_\lambda .$$

From this it follows that  $\mathcal{D}\psi = -\sum_{\lambda=1}^n \theta_\lambda g_\lambda = 0$ . Since  $\bar{\partial}\psi = 0$  and  $\mathcal{D}\psi = 0$ ,  $\psi$  is an element of  $H_{\mathbb{R}}^{0,1}(X)$ . Thus  $\psi = \sum_{\lambda=1}^s c_\lambda d\bar{z}_\lambda + \sum_{\lambda=s+1}^n g_\lambda \bar{\varphi}_\lambda$  for some constant  $c'_\lambda$ s. From (4.1) we conclude that any  $g_\lambda$  is constant. Since  $d\varphi = 0$ , we have  $g_\lambda = 0$  ( $\lambda > r$ ). Accordingly it follows that  $\varphi = \sum_{\lambda=1}^r c_\lambda \varphi_\lambda + \sum_{\lambda=1}^r \overline{c_\lambda} \bar{\varphi}_\lambda$  where  $c'_\lambda$ s are complex numbers. This implies that  $\dim_{\mathbb{R}} H^1(X, \mathbb{R}) = 2r$ , i.e.,  $b_1 = 2r$ .

**Remark 4.1.** In the proof of Theorem 3 we have given an explicit description of elements of  $H_{\mathbb{R}}^{0,1}(X)$ . Since  $\theta$  is isomorphic to  $\mathcal{O}^n$ ,  $H_{\mathbb{R}}^{0,1}(X, \theta)$  is spanned by  $\theta_i \varphi$  ( $i = 1, \dots, n$ ) for elements  $\varphi$  of  $H_{\mathbb{R}}^{0,0}(X)$ .

**Remark 4.2.** If  $X$  is not nilpotent, then  $X = C^r \times F/\Gamma_1$  for a nilpotent manifold  $F$  and a group  $\Gamma_1$  of analytic automorphisms of  $C^s \times F$ . Any element  $g$  of  $\Gamma_1$  induces an automorphism  $g^*$  of  $C^s \times \text{Alb } F$ . Set  $\Gamma_1^* = \{g^* ; g \in \Gamma_1\}$ . Since  $\Gamma_1$  operates on  $C^s \times F$  properly discontinuously without fixed points,  $\Gamma_1^*$  operates on  $C^s \times \text{Alb } F$  in the same way. Thus  $X^* = C^s \times \text{Alb } F/\Gamma_1^*$  is a compact complex manifold, and is therefore parallelisable and solvable. Using this fact we infer that a parallelisable manifold with the following basis  $\{\varphi_i\}$  of  $H^0(X, \Omega^1)$  does not exist:

$$\begin{aligned} d\varphi_1 &= 0, & d\varphi_2 &= \varphi_1 \wedge \varphi_2, & d\varphi_3 &= -2(\mu + 1)\varphi_1 \wedge \varphi_3, \\ d\varphi_4 &= \mu\varphi_1 \wedge \varphi_4, & d\varphi_5 &= (\mu + 1)\varphi_1 \wedge \varphi_5 + \varphi_2 \wedge \varphi_4, \end{aligned}$$

where  $\mu$  is constant, and  $\mu(\mu + 1) \neq 0$ .

*Proof.* If a parallelisable manifold  $X$  of this type exists,  $X^*$  is a parallelisable manifold with a basis  $\{\psi_1, \psi_2, \psi_3, \psi_4\}$  of  $H^0(X^*, \Omega^1)$  such that  $d\psi_1 = 0$ ,  $d\psi_2 = \psi_1 \wedge \psi_2$ ,  $d\psi_3 = -2(\mu + 1)\psi_1 \wedge \psi_3$ ,  $d\psi_4 = \mu\psi_1 \wedge \psi_4$ . This contradicts Lemma 1.4.

### 5. Proof of Theorem 5

First for brevity we assume  $g$  to have the Chevalley decomposition. Let  $\{\varphi_i\}$  and  $\{\theta_i\}$  be dual bases of  $H^0(X, \Omega^1)$  and  $H^0(X, \theta)$  which satisfy (2.1) and (2.2) respectively. The assumption means that  $H_{\mathbb{R}}^{0,1}(X, \theta)$  is generated by  $\theta_i \bar{\varphi}_\lambda$ ,  $\lambda = 1, \dots, n$ , and  $i = 1, \dots, r$ . Define a  $(n - r, n - r)$  matrix  $A = (A_{ij})$  by

$$A_{ij} = 2v_0 \sum_{\lambda < \nu} c_{i+r\lambda} \overline{c_{j+r\lambda}}, \quad i, j = 1, \dots, n - r,$$

where  $v_0 = \int_X dX = i^{-n^2} \int_X \varphi_1 \wedge \dots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \dots \wedge \bar{\varphi}_n$ .

**Lemma 5.1.**  $\det (A_{ij}) \neq 0$ .

*Proof.*  $(\partial\varphi_{i+r}, \partial\varphi_{j+r}) = (d\varphi_{i+r}, d\varphi_{j+r}) = v_0 \sum_{\lambda, \nu} c_{i+r\lambda\nu} \overline{c_{j+r\lambda\nu}} = 2A_{ij}$ . Thus, in order to prove Lemma 5.1, it suffices to show the following:

If for a 1-form  $\psi = \sum_{\lambda=r+1}^n c_\lambda \varphi_\lambda$ ,  $(\partial\psi, \partial\varphi_\nu) = 0$ ,  $\nu = r+1, \dots, n$ , then we have  $\psi = 0$ , where  $c_\lambda$ 's are constant. However this is obvious. q.e.d.

It follows from Lemma 5.1. that there exists  $(n-r, n-r)$  matrix  $(A^{ij})$  such that  $\sum_{i=1}^{n-r} A_{k\lambda} A^{ij} = \delta_{kj}$ .

**Lemma 5.2.** For a  $(0, 2)$ -form  $\varphi = \sum_{\lambda < \nu} a_{\lambda\nu} \bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu$  with some constants  $a_{\lambda\nu}$ ,  $\varphi$  is cohomologous to zero in  $H_2^{0,2}(X)$  if and only if  $\varphi = \sum_{\lambda=r+1}^n a'_\lambda d\bar{\varphi}_\lambda$ , where  $a'_\lambda$  are constants.

*Proof.* For a  $(0, 2)$ -form  $\varphi = \sum_{\lambda < \nu} a_{\lambda\nu} \bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu$  the adjoint operator  $\mathfrak{A}$  of  $\bar{\partial}$  is defined by

$$\mathfrak{A}\varphi = 2 \sum_{\substack{i \\ \lambda < \nu}} c_{i\lambda\nu} a_{\lambda\nu} \bar{\varphi}_i = 2 \sum_{\substack{i \\ \lambda < \nu}} c_{i+r\lambda\nu} a_{\lambda\nu} \bar{\varphi}_{i+r}.$$

Set

$$H\varphi = \varphi - v_0 \sum_{\substack{i, j \\ \lambda < \nu}} A^{ji} c_{i+r\lambda\nu} a_{\lambda\nu} d\bar{\varphi}_{j+r}.$$

If  $\bar{\partial}\varphi = 0$ , then  $H\varphi$  is harmonic, i.e.,  $\bar{\partial}(H\varphi) = 0$ ,  $\mathfrak{A}(H\varphi) = 0$ . In fact,  $\bar{\partial}(H\varphi) = \bar{\partial}\varphi = 0$ . Moreover,

$$\begin{aligned} \mathfrak{A}(H\varphi) &= 2 \sum_{\substack{i \\ \lambda < \nu}} c_{i\lambda\nu} a_{\lambda\nu} \bar{\varphi}_i - 4v_0 \sum_{\substack{i, j, k \\ \lambda < \nu, \alpha < \beta}} c_{i+r\lambda\nu} a_{\lambda\nu} A^{jk} c_{k+r\alpha\beta} \overline{c_{j+r\alpha\beta}} \bar{\varphi}_{k+r} \\ &= 2 \sum_{\substack{i \\ \lambda < \nu}} c_{i\lambda\nu} a_{\lambda\nu} \bar{\varphi}_i - 2 \sum_{\substack{i \\ \lambda < \nu}} c_{i+r\lambda\nu} a_{\lambda\nu} A^{ji} A_{k\lambda\nu} \bar{\varphi}_{k+r} = 0. \end{aligned}$$

Since  $H$  is nothing but the projection of the harmonic part, we have  $H\varphi = 0$  if  $\varphi$  is cohomologous to zero. "If" part of the lemma is obvious.

**Lemma 5.3.** Under some algebraic relations between  $t_{i\lambda}$  ( $i = 1, \dots, n$ , and  $\lambda = 1, \dots, r$ ), there exists a vector  $(0, 1)$ -form  $\psi = \sum_{\alpha=1}^{n_1} \psi_\alpha(t)$  for some  $n_1 \leq n$  such that

$$(5.1) \quad \bar{\partial}\psi - \frac{1}{2}[\psi, \psi] = 0,$$

where  $\psi_1 = \sum_{i=1}^n \sum_{\lambda=1}^r t_{i\lambda} \theta_i \bar{\varphi}_\lambda$ , and  $\psi_\alpha$  is the homogeneous term of total degree  $\alpha$  in  $t_{i\lambda}$ .

*Proof.* Set  $\psi = \sum_{\alpha=1}^{\infty} \psi_\alpha(t)$  and  $\psi_1 = \sum_{i=0}^n \sum_{\lambda=1}^r t_{i\lambda} \theta_i \bar{\varphi}_\lambda$ , where  $\psi_\alpha$  is the homo-

geneous term of total degree  $\alpha$  in  $t_{i\lambda}$ . Since  $[\theta_i\bar{\varphi}_\lambda, \theta_k\bar{\varphi}_\nu] = [\theta_i, \theta_k]\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu$ , we have<sup>1</sup>

$$\bar{\partial}\psi_2 = \frac{1}{2}[\psi_1, \psi_1] - \sum_{i=1}^n \sum_{\substack{\lambda < \nu \\ \lambda, \nu \in Q_1}} a^1_{\lambda\nu i}(t)\theta_i\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu.$$

$\eta_i^1 = \sum_{\lambda < \nu} a^1_{\lambda\nu i}\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu$  is cohomologous to zero in  $H^{0,2}_\partial(X)$ . Hence from Lemma 5.2 it follows<sup>2</sup> that  $\eta_i^1 = \sum_{\mu \in Q_2} b^1_{\mu i}(t)d\bar{\varphi}_\mu$  for some  $b^1_{\mu i}(t)$  and

$$(5.2)_1 \quad a^1_{\lambda\nu i}(t) = 2 \sum_{\mu} b^1_{\mu i}(t)c_{\mu\lambda\nu}.$$

(In general (5.2)<sub>1</sub> is nontrivial; see (3.3).) Then we have

$$\bar{\partial}\psi_3 = [\psi_1, \psi_2] = \sum_{i=0}^2 \sum_{\substack{\lambda \in Q_1 \\ \nu \in Q_2}} a^2_{\lambda\nu i}\theta_i\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu.$$

Again from Lemma 5.2 it follows that

$$\eta_i^2 = \sum_{\substack{\lambda \in Q_1 \\ \nu \in Q_2}} a^2_{\lambda\nu i}\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu = \sum_{\beta \in Q_3} b^2_{\beta i}d\bar{\varphi}_\beta \quad \text{for some } b^2_{\beta i},$$

$$(5.2)_2 \quad a^2_{\lambda\nu i}(t) = 2 \sum_{\mu} b^2_{\mu i}(t)c_{\mu\lambda\nu}.$$

Inductively we define  $\psi_\alpha$  and  $b^{\alpha}_{\mu i}(t)$  under additional relations (5.2)<sub>2</sub>,  $\dots$ , (5.2) <sub>$n-1$</sub> . Since  $\bigcup_{\mu} Q_{\mu}$  is bounded, we obtain the desired  $\psi$  after finite steps of processes. q.e.d.

(5.2)<sub>1</sub>,  $\dots$ , (5.2) <sub>$n-1$</sub>  define an algebraic set  $A$  in  $\mathbb{C}^{nr}$ . Set  $A_\epsilon = \{(t_{i\lambda}) \in A; \sum_{i=0}^2 \sum_{\lambda=1}^n |t_{i\lambda}| < \epsilon\}$  for a sufficiently small positive number  $\epsilon$ . Lemma 5.3 implies that there exists a maximal complex analytic family of deformations of  $X$  depending on  $nr$  parameters  $t_{i\lambda}$ .  $A_\epsilon$  is the Kuranishi space of deformations of  $X$ , [3].

**Lemma 5.4.** *The system of differential equations*

$$(5.3) \quad \bar{\partial}\zeta_\alpha - \psi(t)\zeta_\alpha = 0, \quad \alpha = 1, \dots, n$$

can be solved in  $\mathbb{C}^n \times A_\epsilon$  under the initial condition  $\zeta_\alpha(0) = z_\alpha$ , where  $0 \in A_\epsilon$  denotes the origin of  $\mathbb{C}^{nr}$ .

*Proof.* Since (5.3) is the integrability condition of the system of differential equations (5.3), we can formally solve it by the iteration method. To this end we must show that the formal solutions converge in  $\mathbb{C}^n \times A_\epsilon$  for a sufficient small positive number  $\epsilon$ .

In view of Propositions 2.3 and 2.4 together with Lemma 5.3,  $\psi$  is represented by

<sup>1,2</sup> See (1.4) as for  $Q_1, Q_2$ , etc.

$$\psi = \sum_{i=1}^n \sum_{l=1}^r t_{il} \theta_i \bar{\varphi}_l + \sum_{k=r+1}^n \sum_{\mu=r+1}^m \sum_{\beta=2}^{n_1} a_{k\mu}^\beta(t) \theta_k \bar{\varphi}_\mu,$$

where  $m = \#Q$  (see (1.2)), and  $a_{k\mu}^\beta$  denotes the homogeneous term of total degree  $\beta$  in  $t_{i\lambda}$ . Then we have  $\sum_{\mu=r+1}^m a_{k\mu}^\beta \bar{\varphi}_\mu = \sum_{\mu=r+1}^m b_{k\mu}^\beta(t, \bar{z}) d\bar{z}_\mu$  for a polynomial  $b_{k\mu}^\beta(t, \bar{z})$  in  $\bar{z}_1, \dots, \bar{z}_{m-1}, t_{i\lambda}$ , which is of degree  $\beta$  in  $t_{i\lambda}$ . Therefore the system of differential equations (5.3) is equivalent to the system of equations:

$$\begin{aligned} \bar{\partial}_1 \zeta &= \sum_{i=1}^n t_{i1} \theta_i \zeta + \sum_{\beta=2}^{n_1} \sum_{i=r+1}^n b_{i1}^\beta(t, \bar{z}) \theta_i \zeta, \\ &\dots\dots\dots \\ \bar{\partial}_r \zeta &= \sum_{i=1}^n t_{ir} \theta_i \zeta + \sum_{\beta=2}^{n_1} \sum_{i=r+1}^n b_{i1}^\beta(t, \bar{z}) \theta_i \zeta, \\ \bar{\partial}_\mu \zeta &= \sum_{\beta=l}^{n_1} \sum_{i=r+1}^n b_{i\mu}^\beta \theta_i \zeta \quad \text{for } \mu \in Q (l \geq 2), \\ &= 0 \quad \text{for } \mu \in Q_\infty. \end{aligned} \tag{5.4}$$

Set  $\zeta = \sum_{\beta=0}^\infty \zeta_\beta$  where  $\zeta_\beta$  denotes the homogeneous term of total degree  $\beta$  in  $t_{i\lambda}$ .

Case 1. Assume  $\zeta(0) = z_\alpha$  or equivalently  $\zeta_0 = z_\alpha$  ( $\alpha = 1, \dots, r$ ). Then we have

$$\bar{\partial}_\mu \zeta_1 = t_{\alpha\mu}, \quad \mu \leq r; \quad \bar{\partial}_\mu \zeta_1 = 0, \quad \mu > r.$$

Hence setting  $\zeta_1 = \sum_{\mu=1}^r t_{\alpha\mu} \bar{z}_\mu$ , we obtain the solution  $\zeta = z_\alpha + \sum_{\mu=1}^r t_{\alpha\mu} \bar{z}_\mu$ .

Case 2. Assume  $\zeta_0 = z_\alpha$  ( $\alpha \in Q$ ). Denote by  $D_i(f)$  the degree of a polynomial  $f$  with respect to  $z_i$ . Since  $\bar{\partial}_\mu \zeta_1 = t_{\alpha\mu}$  ( $\mu \leq r$ ),  $\bar{\partial}_\mu \zeta_1 = 0$  ( $\mu > r$ ), we have

$$\zeta_1 = \sum_{\mu=1}^r t_{\alpha\mu} \bar{z}_\mu. \text{ From Proposition 2.4 it follows that for } \mu (\leq m)$$

$$\theta_1 z = 0 \ (i > \mu \text{ or } i \leq s), \quad \theta_\mu z_\mu = 1, \quad \theta_i z_\mu = G_{i\mu} \ (\mu > i > s),$$

where  $G_{i\mu}$  is a polynomial in  $z_1, \dots, z_{i-1}$ . Hence we have  $(D_\alpha(\zeta_2) = 0, D_\gamma(\zeta_2) = 0 \ (\gamma > \alpha)$ . If  $D_{\alpha-1}(\zeta_2) = N$ , then  $D_{\alpha-1}(\zeta_3) = N - 1, D_\gamma(\zeta_3) = 0 \ (\gamma \geq \alpha)$ . Inductively we obtain  $D_{\alpha-1}(\zeta_{N+2}) = 0, D_\gamma(\zeta_{N+2}) = 0 \ (\gamma \geq \alpha)$ . For a sufficiently large integer  $N_1$  we have

$$D_\gamma(\zeta_{N_1+\delta}) = 0, \quad \gamma = 1, \dots, n, \quad \delta = 1, \dots, n_1,$$

so that we may set  $\zeta_\beta = 0$  for any  $\beta > N_1 + \delta$ . Hence  $\zeta = \sum_{\beta=0}^{N_1+\delta} \zeta_\beta$  is the desired solution.

Case 3. Assume  $\zeta_0 = z_\alpha$  ( $\alpha \in Q_\infty$ , i.e.,  $\alpha \geq m + 1$ ). Similarly, as in Case 2 we have  $D_\gamma(\zeta_{N+\delta}) = 0$ ,  $\gamma = s + 1, \dots, n$ ,  $\delta = 1, \dots, n_1$ . Therefore the problem is reduced to the case where

$$r = s, \text{ and } \zeta_0 \text{ is a polynomial in } e^{z_1}, z_1, \dots, z_r,$$

and it suffices to prove the convergence of the series  $\sum_{\beta=0}^\infty \zeta_\beta$  only for  $\zeta_0 = e^{z_1} z_1^{e_1} \dots z_r^{e_r}$ . Moreover, the system of differential equations (5.4) takes the form:

$$\bar{\partial}_\mu \zeta = \sum_{\lambda=1}^r t_{\mu\lambda} \partial_\lambda \zeta.$$

Define the norms  $\| \cdot \|$  by

$$\|z\| = \sum_{i=1}^r |z_i|, \quad \|t\| = \sum_{i=1}^n \sum_{\lambda=1}^r |t_{i\lambda}|, \quad \|f\| = \sum_{\beta_1, \dots, \beta_r} |a_{\beta_1} \dots \beta_r| |z_1^{\beta_1} \dots z_r^{\beta_r}|$$

for a polynomial  $f = \sum a_{\beta_1 \dots \beta_r} z_1^{\beta_1} \dots z_r^{\beta_r}$ . Since  $\bar{\partial}_\mu \zeta_1 = \sum t_{\mu\lambda} \partial_\lambda \zeta_0$ , we have

$$\zeta_1 = \sum t_{\mu\lambda} (\partial_\lambda \zeta_0) \bar{z}_\mu,$$

and therefore  $|\zeta_1| \leq \|t\| \|z\| \|\zeta_0\|$ ,  $\bar{\partial}_\mu \zeta_2 = \sum t_{\mu\lambda} \partial_\lambda \zeta_1$ . Thus  $|\zeta_2| \leq \frac{\|t\|^2 \|z\|^2}{2!} \|\zeta_0\|$ .

Inductively we have  $|\zeta_k| \leq \frac{\|t\|^k \|z\|^k}{k!} \|\zeta_0^{(k)}\|$ , and

$$\zeta_0^{(k)} = \sum_{k_0 + \dots + k_r = k} \frac{k!}{k_0! k_1! \dots k_r!} e^{z_1(z_1^{e_1})^{k_1} \dots (z_r^{e_r})^{k_r}}.$$

Therefore  $\|\zeta_0^{(k)}\| \leq \frac{k!}{(k - e_0)!} M \cdot G(z)$  where  $e_0 = \sum_{\nu=1}^r e_\nu$ ,  $G(z) = |e^{z_1}| \cdot \sum_{0 \leq j_\nu \leq e_\nu} |z_1^{j_1} |z_2^{j_2} \dots |z_r^{j_r}|$ , and  $M$  is a sufficiently large positive number independent of  $k$ . Hence

$$\begin{aligned} \left| \sum_{k=0}^\infty \zeta_k \right| &\leq \left| \sum_{k=0}^{e_0-1} \zeta_k \right| + \left| \sum_{k=e_0}^\infty \zeta_k \right| \\ &\leq \left| \sum_{k=0}^{e_0-1} \zeta_k \right| + M \cdot G(z) (\|t\| \|z\|)^{e_0} \sum_{k=0}^\infty \frac{1}{k!} \|t\|^k \|z\|^k \\ &= \left| \sum_{k=0}^{e_0-1} \zeta_k \right| + M \cdot G(z) (\|t\| \|z\|)^{e_0} \exp(\|t\| \|z\|). \quad \text{q.e.d.} \end{aligned}$$

In view of the proof of Lemma 5.4 we have

$$d\zeta_1 \wedge \dots \wedge d\zeta_n \wedge d\bar{\zeta}_1 \wedge \dots \wedge d\bar{\zeta}_n$$

$$= c(t)dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n,$$

where a differentiable function  $c(t)$  of  $t_{i\lambda}$  is independent of  $z_1, \dots, z_n$  and satisfies  $c(0) = 1$ . Hence by the same argument as in §3 we conclude that any small deformation  $X_t$  ( $t \in A_\varepsilon$ ) has  $\mathbb{C}^n$  as the universal covering for a sufficiently small positive number  $\varepsilon$ .

In the general case we can apply the following lemma which is a weaker form of Propositions 2.2 and 2.3.

**Lemma.** *Let  $G$  be a connected solvable complex Lie group. Then we can choose a global coordinate  $(z_1, \dots, z_n)$  of  $G$  ( $\cong \mathbb{C}^n$ ) and a basis  $\{\varphi_\lambda\}, \{\theta_\lambda\}$  of right invariant 1-forms and vector fields respectively such that*

$$\varphi_\lambda = \sum_{\nu=1}^{\lambda} F_{\lambda\nu}(z)dz_\nu, \quad \theta_\lambda = \sum_{\nu=\lambda}^n G_{\lambda\nu}(z)\partial/\partial z_\nu,$$

where  $F_{\lambda\nu} = F_{\lambda\nu}(z_1, \dots, z_{\nu-1})$ ,  $G_{\lambda\nu} = G_{\lambda\nu}(z_1, \dots, z_{\nu-1})$  and  $(\theta_\lambda, \varphi_\nu) = \delta_{\lambda\nu}$ .

By quite similar arguments we can also prove Theorem 5.

## 6. Classification of four- and five-dimensional complex solvable manifolds

By an elementary calculation together with Lemma 1.4 and Remark 4.2 we classify four and five-dimensional complex solvable Lie groups which may have uniform subgroups as follows:

Type IV:

1.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 4$ .
2.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 3$ ,  $d\varphi = -\varphi_2 \wedge \varphi_3$ .
3.  $d\varphi_1 = 0$ ,  $d\varphi_2 = 0$ ,  $d\varphi_3 = -\varphi_1 \wedge \varphi_2$ ,  $d\varphi_4 = -2\varphi_1 \wedge \varphi_3$ .
4.  $d\varphi_1 = 0$ ,  $d\varphi_2 = 0$ ,  $d\varphi_3 = \varphi_2 \wedge \varphi_3$ ,  $d\varphi_4 = -\varphi_2 \wedge \varphi_4$ .
5.  $d\varphi_1 = 0$ ,  $d\varphi_2 = \varphi_1 \wedge \varphi_2$ ,  $d\varphi_3 = \alpha\varphi_1 \wedge \varphi_3$ ,  
 $d\varphi_4 = -(1 + \alpha)\varphi_1 \wedge \varphi_4$ ,  $\alpha(1 + \alpha) \neq 0$ .
6.  $d\varphi_1 = 0$ ,  $d\varphi_2 = \varphi_1 \wedge \varphi_2$ ,  $d\varphi_3 = -\varphi_1 \wedge \varphi_3$ ,  $d\varphi_4 = -\varphi_2 \wedge \varphi_3$ .
7.  $d\varphi_1 = 0$ ,  $d\varphi_2 = \varphi_1 \wedge \varphi_2$ ,  $d\varphi_3 = -2\varphi_1 \wedge \varphi_3$ ,  
 $d\varphi_4 = d\varphi_1 \wedge \varphi_4 - \varphi_1 \wedge \varphi_2$ .

Type V:

1.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 5$ .
2.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 4$ ,  $d\varphi_5 = -\varphi_3 \wedge \varphi_4$ .
3.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 4$ ,  $d\varphi_5 = -\varphi_1 \wedge \varphi_3 - \varphi_2 \wedge \varphi_4$ .
4.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 3$ ,  $d\varphi_4 = -\varphi_1 \wedge \varphi_2$ ,  $d\varphi_5 = -\varphi_1 \wedge \varphi_3$ .
5.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 3$ ,  $d\varphi_4 = -\varphi_2 \wedge \varphi_3$ ,  $d\varphi_5 = -2\varphi_2 \wedge \varphi_4$ .



6.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 3$ ,  $d\varphi_4 = -\varphi_1 \wedge \varphi_2$ ,  
 $d\varphi_5 = -2\varphi_1 \wedge \varphi_4 - \varphi_2 \wedge \varphi_3$ .
7.  $d\varphi_\lambda = 0$ ,  $1 \leq \lambda \leq 3$ ,  $d\varphi_4 = \varphi_3 \wedge \varphi_4$ ,  $d\varphi_5 = -\varphi_3 \wedge \varphi_5$ .
8.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = -\varphi_1 \wedge \varphi_2$ ,  $d\varphi_4 = -2\varphi_1 \wedge \varphi_3$ ,  
 $d\varphi_5 = -2\varphi_2 \wedge \varphi_3$ .
9.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = -\varphi_1 \wedge \varphi_2$ ,  $d\varphi_4 = -2\varphi_1 \wedge \varphi_3$ ,  
 $d\varphi_5 = -3\varphi_1 \wedge \varphi_4$ .
10.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = -\varphi_1 \wedge \varphi_2$ ,  $d\varphi_4 = -2\varphi_1 \wedge \varphi_3$ ,  
 $d\varphi_5 = -3\varphi_1 \wedge \varphi_4 - \varphi_2 \wedge \varphi_3$ .
11.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = -\varphi_1 \wedge \varphi_2$ ,  $d\varphi_4 = \varphi_1 \wedge \varphi_4$ ,  
 $d\varphi_5 = \varphi_1 \wedge \varphi_5$ .
12.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = \varphi_1 \wedge \varphi_3$ ,  $d\varphi_4 = \varphi_2 \wedge \varphi_4$ ,  
 $d\varphi_5 = -(\varphi_1 + \varphi_2) \wedge \varphi_5$ .
13.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = \varphi_2 \wedge \varphi_3$ ,  $d\varphi_4 = \alpha\varphi_2 \wedge \varphi_4$ ,  
 $d\varphi_5 = -(1 + \alpha)\varphi_2 \wedge \varphi_5$ ,  $\alpha(1 + \alpha) \neq 0$ .
14.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = \varphi_1 \wedge \varphi_3$ ,  $d\varphi_4 = -2\varphi_1 \wedge \varphi_4$ ,  
 $d\varphi_5 = \varphi_1 \wedge \varphi_5 - \varphi_1 \wedge \varphi_3$ .
15.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = \varphi_2 \wedge \varphi_3$ ,  $d\varphi_4 = -\varphi_2 \wedge \varphi_4$ ,  
 $d\varphi_5 = -\varphi_3 \wedge \varphi_4$ .
16.  $d\varphi_\lambda = 0$ ,  $\lambda = 1, 2$ ,  $d\varphi_3 = \varphi_1 \wedge \varphi_3$ ,  $d\varphi_4 = -\varphi_1 \wedge \varphi_4$ ,  
 $d\varphi_5 = -\varphi_3 \wedge \varphi_4 - \varphi_1 \wedge \varphi_2$ .
17.  $d\varphi_1 = 0$ ,  $d\varphi_2 = \varphi_1 \wedge \varphi_2$ ,  $d\varphi_3 = \alpha\varphi_1 \wedge \varphi_3$ ,  $d\varphi_4 = \beta\varphi_1 \wedge \varphi_4$ ,  
 $d\varphi_5 = -(1 + \alpha + \beta)\varphi_1 \wedge \varphi_5$ ,  $\alpha\beta(1 + \alpha + \beta) \neq 0$ .
18.  $d\varphi_1 = 0$ ,  $d\varphi_2 = -3\varphi_1 \wedge \varphi_2$ ,  $d\varphi_3 = \varphi_1 \wedge \varphi_3$ ,  
 $d\varphi_4 = \varphi_1 \wedge \varphi_4 - \varphi_1 \wedge \varphi_3$ ,  $d\varphi_5 = \varphi_1 \wedge \varphi_5 - \varphi_1 \wedge \varphi_3$ .
19.  $d\varphi_1 = 0$ ,  $d\varphi_2 = \varphi_1 \wedge \varphi_2$ ,  $d\varphi_3 = -\varphi_1 \wedge \varphi_3$ ,  
 $d\varphi_4 = \varphi_1 \wedge \varphi_4 - \varphi_1 \wedge \varphi_2$ ,  $d\varphi_5 = -\varphi_1 \wedge \varphi_5 - \varphi_1 \wedge \varphi_3$ .
20.  $d\varphi_1 = 0$ ,  $d\varphi_2 = \varphi_1 \wedge \varphi_2$ ,  $d\varphi_3 = \varphi_1 \wedge \varphi_3 - \varphi_1 \wedge \varphi_2$ ,  
 $d\varphi_4 = \alpha\varphi_1 \wedge \varphi_4$ ,  $d\varphi_5 = -(2 + \alpha)\varphi_1 \wedge \varphi_5$ ,  $\alpha(2 + \alpha) \neq 0$ .

**Lemma 6.1.** *Let  $A$  be a  $3 \times 3$  matrix which induces an automorphism of a complex torus of dimension 3. Assume that  $A$  has eigenvalues  $\alpha, \alpha$  and  $\alpha^{-2}$ . Then  $\alpha$  is a root of unity.*

*Proof.* Let  $\Phi$  be the proper polynomial of  $A$ . Then

$$\Psi(x) = \Phi(x)\overline{\Phi(x)} \in Z[x].$$

Assume  $\alpha$  is not a root of 1. We shall prove  $\Psi$  is irreducible in  $Z[x]$ . In fact, if  $\Psi$  is not irreducible, there exist  $\Phi_1, \Phi_2 \in Z[x]$  such that  $\Psi = \Phi_1\Phi_2$ . We may assume  $\deg \Phi_1 \geq \deg \Phi_2 \geq 2$ . Two cases may occur. First, we assume  $\deg \Phi_1 = 4, \deg \Phi_2 = 2$ . Put  $F(x) = (x - \alpha)(x - \bar{\alpha}) \in R[x]$ . If  $F | \Phi_2$  in  $R[x]$ , then  $F = \pm \Phi_2$ . Since the constant term of  $\Psi$  equals  $\pm 1$ , that of  $\Phi_2$  equals  $\pm 1$ . Hence  $|\alpha| = 1$ . Any conjugate of  $\alpha$  has absolute value 1. Therefore  $\alpha$  is a root of 1. This is a contradiction. If  $F \nmid \Phi_2$  in  $R[x]$ , then  $F^2 | \Phi_1$  in  $R[x]$ , hence  $F^2 = \pm \Phi_1$ . Similarly, we are led to the contradiction. In case  $\deg \Phi_1 = \deg \Phi_2 = 3$ , we also have a contradiction. Thus  $\Psi$  is proved to be irreducible. This contradicts the fact that  $\Psi$  has a double root. q.e.d.

Similarly we obtain

**Lemma 6.2.** *Let  $A$  be a  $4 \times 4$  matrix which induces an automorphism of a 4-dimensional complex torus. Assume  $A$  has eigenvalues  $\alpha, \alpha, \alpha, \alpha^{-3}$ . Then  $\alpha$  is a root of 1.*

From these lemmas, we conclude that a parallelisable manifold of type IV-6, V-14 or V-18 does not exist. In fact, in the case of IV-7 we consider the Mostow decomposition  $\pi : X \rightarrow B$ . Then  $B$  is an elliptic curve, the fiber  $F$  is a complex torus of dimension 3,

$$X = C^4 / \Gamma, \quad g = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Gamma,$$

and  $g : F \rightarrow F$  is given by

$$(z_2, z_3, z_4) \mapsto (e^{-\omega_1}z_2 + \omega_2, e^{2\omega_1}z_3 + \omega_3, e^{-\omega_1}z_4 + e^{-\omega_1}\omega_1z_2 + \omega_4).$$

Lemma 6.1 shows that  $e^{\omega_1}$  is a root of 1. This contradicts the fact that  $\{\omega_1; (\omega_1, \omega_2, \omega_3, \omega_4) \in \Gamma\}$  are periods of the elliptic curve  $B$ .

Similarly it can be proved that a parallelisable manifold of type V-15 or V-18 does not exist.

The author does not know whether there exist parallelisable manifolds of types IV-5, V-11, V-13, V-16, V-19, V-20. In other cases we can construct examples of each type. Now we summarize the results. In the following table we omit  $z_i + y_i$  for simplicity. For example,  $z * y = (z_3 + y_3 + y_1z_2)$  implies  $z * y = (z_1 + y_1, z_2 + y_2, z_3 + y_3 + y_1z_2)$ .

Type IV	$z * y$	
1	abelian	
2	nilpotent	$z_4 + y_4 + y_2z_3$
3	nilpotent	$z_3 + y_3 + y_1z_2, z_4 + y_4 + 2y_1z_3 + y_1^2z_2$
4	solvable	$e^{-y_2}z_3 + y_3, e^{y_2}z_4 + y_4$
5	solvable	$e^{-y_1}z_2 + y_2, e^{-\alpha y_1}z_3 + y_3, e^{(1+\alpha)y_1}z_4 + y_4$
6	solvable	$e^{-y_1}z_2 + y_2, e^{y_1}z_3 + y_3, z_4 + y_4 + e^{y_1}y_2z_3$

Type	V	$z * y$
1	abelian	
2	nilpotent	$z_5 + y_5 + y_3 z_4$
3	nilpotent	$z_5 + y_5 + y_1 z_3 + y_2 z_4$
4	nilpotent	$z_4 + y_4 + y_1 z_2, z_5 + y_5 + y_1 z_3$
5	nilpotent	$z_4 + y_4 + y_2 z_3, z_5 + y_5 + 2y_2 z_4 + y_3^2 z_3$
6	nilpotent	$z_4 + y_4 + y_1 z_2, z_5 + y_5 + 2y_1 z_4 + y_2 z_3 + y_1^2 z_2$
7	solvable	$e^{-y_2 z_4} + y_4, e^{y_2 z_5} + y_5$
8	nilpotent	$z_3 + y_3 + y_1 z_2, z_4 + y_4 + 2y_1 z_3 + y_1^2 z_2,$ $z_5 + y_5 + 2y_2 z_3 + y_1 z_3^2 + 2y_1 y_2 z_2$
9	nilpotent	$z_3 + y_3 + y_1 z_2, z_4 + y_4 + 2y_1 z_3 + y_1^2 z_2,$ $z_5 + y_5 + 3y_1 z_4 + 3y_1^2 z_3 + y_1^3 z_2$
10	nilpotent	$z_3 + y_3 + y_1 z_2, z_4 + y_4 + 2y_1 z_3 + y_1^2 z_2,$ $z_5 + y_5 + 3y_1 z_4 + (3y_1^2 + 2y_2) z_3 + y_1 z_3^2 + (y_1^3 + 2y_1 y_2) z_2$
11	solvable	$z_3 + y_3 + y_1 z_2, e^{-y_1 z_4} + y_4, e^{y_1 z_5} + y_5$
12	solvable	$e^{-y_1 z_3} + y_3, e^{-y_2 z_4} + y_4, e^{y_1 + y_2 z_5} + y_5$
13	solvable	$e^{-y_2 z_3} + y_3, e^{-\alpha y_2 z_4} + y_4, e^{(1+\alpha)y_2 z_5} + y_5$
15	solvable	$e^{-y_2 z_3} + y_3, e^{y_2 z_4} + y_4, z_5 + y_5 + e^{y_2} y_3 z_4$
16	solvable	$e^{-y_1 z_3} + y_3, e^{y_1 z_4} + y_4, z_5 + y_5 + e^{y_1} y_3 z_4 + y_1 z_2$
17	solvable	$e^{-y_1 z_2} + y_2, e^{-\alpha y_1 z_3} + y_3, e^{-\beta y_1 z_4} + y_4, e^{(1+\alpha+\beta)z_5} + y_5$
19	solvable	$e^{-y_1 z_2}, e^{y_1 z_3} + y_3, e^{-y_1 z_4} + e^{-y_1 y_1 z_2}, e^{y_1 z_5} + y_5 + e^{y_1} y_1 z_3$
20	solvable	$e^{-y_1 z_2} + y_2, e^{-y_1 z_3} + y_3 + e^{-y_1} y_1 z_2, e^{-\alpha y_1 z_4} + y_4,$ $e^{-(\alpha+2)y_1 z_5} + y_5$

Complex solvable manifolds of dimensions 4, 5 are classified as follows :

	$r$	$h^{0,1}$	structure (Albanese mapping)
IV: 1	4	4	complex torus
2	3	3	$T^1$ -bundle over $T^3$
3	2	2	$T^2$ -bundle over $T^2$
4	2	2,4	$T^2$ -bundle over $T^2$
5	1	1,2,4	$T^3$ -bundle over $T^1?$
6	1	1,3	(III-2)-bundle over $T^1$
V: 1	5	5	complex torus
2	4	4	$T^1$ -bundle over $T^4$
3	4	4	$T^1$ -bundle over $T^4$
4	3	3	$T^2$ -bundle over $T^3$
5	3	3	$T^2$ -bundle over $T^3$
6	3	3	$T^2$ -bundle over $T^3$
7	3	3,5	$T^2$ -bundle over $T^3$
8	2	2	$T^3$ -bundle over $T^2$

	$r$	$h^{0,1}$	structure (Albanese mapping)
9	2	2	$T^3$ -bundle over $T^2$
10	2	2	$T^3$ -bundle over $T^2$
11	2	2,4	$T^3$ -bundle over $T^2?$
12	2	2,3,5	$T^3$ -bundle over $T^2$
13	2	2,3,5	$T^3$ -bundle over $T^2?$
15	2	2,4	(III-2)-bundle over $T^2$
16	2	2,4	(III-2)-bundle over $T^2?$
17	1	1,2,3,5	$T^4$ -bundle over $T^1$
19	1	1,3	$T^4$ -bundle over $T^1?$
20	1	1,2,4	$T^4$ -bundle over $T^1?$

Here  $r = \dim H^0(X, d\mathcal{O})$ ,  $h^{0,1} = \dim H^1(X, \mathcal{O})$ ,  $T^n =$  a complex torus of dimension  $n$ .

**Remark.** A solvable manifold of dimension 4 or 5 has a Lie algebra with the Chevalley decomposition, and so from Theorem 3 it follows that  $b_1 = 2r$ .

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